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# FOURIER-MUKAI TRANSFORMS AND BRIDGELAND STABILITY CONDITIONS ON ABELIAN THREEFOLDS II

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**ABSTRACT.** We show that the conjectural construction proposed by Bayer, Bertram, Macrì and Toda gives rise to Bridgeland stability conditions for a principally polarized abelian threefold with Picard rank one by proving that tilt stable objects satisfy the strong Bogomolov-Gieseker type inequality. This is done by showing certain Fourier-Mukai transforms give equivalences of abelian categories which are double tilts of coherent sheaves.

## INTRODUCTION

There is a growing interest in the study of Bridgeland stability conditions on varieties. This notion was introduced in [Bri1] and some known examples can be found in [Bri2, AB, Oka, Mac1, Mac2, MP, Sch]. See [Huy3] and [BBR, Appendix D] for comprehensive expositions on the subject. Construction of such stability conditions on a given Calabi-Yau threefold is an important problem but the only known example is on an abelian threefold (see [MP]). For further motivation from, for example, Mathematical Physics, see [Clay, Tod1]. A conjectural construction of such a stability condition for any projective threefold was introduced by Bayer, Bertram, Macrì and Toda in [BMT, BBMT]. Here they introduced the notion of tilt stability for objects in an abelian subcategory of the derived category which is a tilt of coherent sheaves. These have now been studied extensively: [Tod2, LM, BMT, Mac2, MP, Sch]. This conjectural construction has boiled down to the requirement that certain tilt stable objects satisfy a so-called (weak) Bogomolov-Gieseker (B-G for short) type inequality. Moreover they went in to propose a stronger version of this inequality which is known to hold for projective 3-space (see [Mac2]) and smooth quadric threefold (see [Sch]). In [MP], for a principally polarized abelian threefold, we prove that tilt stable objects satisfy the weak B-G type inequality associated to a special complexified ample class. It was achieved by establishing an equivalence of two abelian categories given by the classical Fourier-Mukai transform with kernel the Poincaré bundle. The aim of this paper is to extend those ideas for certain kind of non-trivial Fourier-Mukai transforms (FMT for short) to establish the *strong* B-G type inequality for the same abelian threefold.

For an abelian variety  $X$ , the group of FMTs  $\text{Aut } D^b(X)$  is well understood via the notion of isometric isomorphism (see [Orl2] or [Huy1, Chapter 9]). To any FMT  $\Phi_{\mathcal{E}}$  with kernel  $\mathcal{E}$  Orlov constructed an isometric automorphism  $f_{\mathcal{E}}$  of the product  $X \times \widehat{X}$ . He showed that  $\Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}$  is a surjective map of groups and its kernel consists of trivial FMTs (which send skyscraper sheaves to skyscraper sheaves up to shift) which also preserve  $\mathbf{Pic}^0(X)$  up to shift.

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When  $(X, L)$  is a principally polarized abelian variety, let  $\widetilde{\mathrm{SL}(2, \mathbb{Z})}$  be the central  $\mathbb{Z}$ -extension of the group  $\mathrm{SL}(2, \mathbb{Z})$  generated by the FMT with kernel the Poincaré bundle on the product  $X \times X$ ,  $(-) \otimes L$  and  $[1]$  as a subgroup of  $\mathrm{Aut} D^b(X)$ . So  $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$  is a subgroup of  $\mathrm{Aut} D^b(X)$  and one can canonically identify the isometric automorphisms of it with elements from  $\mathrm{SL}(2, \mathbb{Z})$ . Any FMT  $\Phi_{\mathcal{E}}$  induces a linear isomorphism  $\Phi_{\mathcal{E}}^H \in \mathrm{GL}(H^{2*}(X, \mathbb{Q}))$ , called the cohomological Fourier-Mukai transform, and this gives rise to a representation of the corresponding group of FMTs. In this paper, when  $X$  is principally polarized with Picard rank one, we obtain an explicit matrix description for this representation of  $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$  in terms of  $f_{\mathcal{E}}$ . As a result, any such induced non-trivial transform on  $H^{2*}(X, \mathbb{Q})$  is an anti-diagonal matrix with respect to some suitable twisted Chern characters. This allows us to handle the numerology in the same way as that of the classical FMT. A matrix representation for the induced transform of an abelian surface was also considered in [YY].

When  $\alpha \in \mathrm{NS}_{\mathbb{C}}(X)$  is a complexified ample class, it is expected that  $Z_{\alpha}(-) = -\int_X e^{-\alpha} \mathrm{ch}(-)$  defines a central charge function of some stability condition on  $X$ . The space of all stability conditions carries a natural left action of the group  $\mathrm{Aut} D^b(X)$ . This can be defined via a natural left action of  $\mathrm{Aut} D^b(X)$  on  $\mathrm{Hom}(K(X), \mathbb{C})$ . When  $(X, L)$  is a  $g$ -dimensional principally polarized abelian variety with Picard rank one, we can view the action of  $\Phi_{\mathcal{E}} \in (X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$  on  $Z_{\alpha}$  explicitly as:  $\Phi_{\mathcal{E}} \cdot Z_{\alpha} = \zeta Z_{\alpha'}$  for some  $\alpha' \in \mathrm{NS}_{\mathbb{C}}(X)$  and  $\zeta \in \mathbb{C}^*$  (see [MY, Appendix] for the dimension 2 case). When  $\zeta$  is real one can expect that the FMT  $\Phi_{\mathcal{E}}$  gives an equivalence of some hearts of particular stability conditions of  $D^b(X)$  whose  $\alpha$  and  $\alpha'$  are determined by  $\Im \zeta = 0$  (see Note 3.2). For example when  $g = 2$ , following similar ideas in [Yos], one can show that any FMT gives an equivalence of two abelian categories each of which are tilts of  $\mathrm{Coh}(X)$  (see [Huy2]). Understanding the homological FMT for the case of  $g = 3$  is the basis of this paper. When the Picard rank is 1, this amounts to understanding the transforms as a numerical matrix. This then allows us, in a similar way to [MP], to show that any non-trivial FMT in  $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$  gives an equivalence of two abelian categories each of which are double tilts of  $\mathrm{Coh}(X)$  (see Theorem 4.6). Minimal objects are sent to minimal objects again under an FMT. This enables us to obtain an inequality involving the top part of the Chern character of minimal objects in these abelian categories, and this is exactly the strong B-G type inequality. Therefore any tilt stable object with zero tilt slope satisfies the strong inequality in [BMT] for our abelian threefold case (see Theorem 4.8).

Toda considered similar ideas in an attempt to construct a ‘‘Gepner’’ type stability condition on a quintic threefold using the spherical twist of the structure sheaf (see [Tod3]). In [Pol], Polishchuk tested the existence of stability conditions for abelian varieties by studying ‘‘Lagrangian-invariant’’ objects of  $D^b(X)$ .

## NOTATION

We follow the notation of the first paper [MP] which we summarize and extend as follows.

- (i) We will denote an  $n \times n$  anti-diagonal matrix with entries  $a_k$  by

$$\mathrm{adiag}(a_1, \dots, a_n)_{ij} := \begin{cases} a_k & \text{if } i = k, j = n + 1 - k \\ 0 & \text{otherwise.} \end{cases}$$

- (ii) For  $0 \leq i \leq \dim X$ ,  $\text{Coh}^{\leq i}(X) := \{E \in \text{Coh}(X) : \dim \text{Supp}(E) \leq i\}$ ,  $\text{Coh}^{\geq i}(X) := \{E \in \text{Coh}(X) : \text{for } 0 \neq F \subset E, \dim \text{Supp}(F) \geq i\}$  and  $\text{Coh}^i(X) := \text{Coh}^{\leq i}(X) \cap \text{Coh}^{\geq i}(X)$ .
- (iii) For an interval  $I \subset \mathbb{R} \cup \{+\infty\}$ ,  $\text{HN}_{\omega,B}^\mu(I) := \{E \in \text{Coh}(X) : [\mu_{\omega,B}^-(E), \mu_{\omega,B}^+(E)] \subset I\}$ . Similarly the subcategory  $\text{HN}_{\omega,B}^\nu(I) \subset \mathcal{B}_{\omega,B}$  is defined.
- (iv) For a Fourier-Mukai functor  $\Upsilon$  and a heart  $\mathfrak{A}$  of a t-structure for which  $D^b(X) \cong D^b(\mathfrak{A})$ ,  $\Upsilon_{\mathfrak{A}}^k(E) := H_{\mathfrak{A}}^k(\Upsilon(E))$ . For a sequence of integers  $i_1, \dots, i_s$ ,

$$V_{\mathfrak{A}}^\Upsilon(i_1, \dots, i_s) := \{E \in D^b(X) : \Upsilon_{\mathfrak{A}}^j(E) = 0 \text{ for } j \notin \{i_1, \dots, i_s\}\}.$$

If  $\Upsilon$  is a Fourier-Mukai transform then  $E \in \text{Coh}(X)$  being  $\Upsilon$ -WIT $_i$  is equivalent to  $E \in V_{\text{Coh}(X)}^\Upsilon(i)$ .

- (v) For a  $g$ -dimensional polarized projective variety  $(X, L)$  with Picard rank one over  $\mathbb{C}$ , the Chern character of any  $E \in D^b(X)$  is of the form  $(a_0, a_1\ell, a_2\ell^2/2!, \dots, a_g\ell^g/g!)$  for some  $a_i \in \mathbb{Z}$ . Here  $\ell := c_1(L)$ . For simplicity we write  $\text{ch}(E) = (a_0, a_1, a_2, \dots, a_g)$ . Also we abuse notation to write  $L^k$  for the functor  $(-) \otimes L^k$ .

## 1. PRELIMINARIES

**1.1. Construction of stability conditions for threefolds.** Let us quickly recall the conjectural construction of stability conditions for a given smooth projective threefold  $X$  over  $\mathbb{C}$  as introduced in [BMT]. Let  $\omega, B$  be in  $\text{NS}_{\mathbb{R}}(X)$  with  $\omega$  an ample class, i.e.  $B + i\omega \in \text{NS}_{\mathbb{C}}(X)$  is a complexified ample class. The twisted Chern character with respect to  $B$  is defined by  $\text{ch}^B(-) = e^{-B} \text{ch}(-)$ . The twisted slope  $\mu_{\omega,B}(E)$  of  $E \in \text{Coh}(X)$  is defined by

$$\mu_{\omega,B}(E) = \begin{cases} +\infty & \text{if } E \text{ is a torsion sheaf} \\ \frac{\omega^2 \text{ch}_1^B(E)}{\text{ch}_0^B(E)} & \text{otherwise.} \end{cases}$$

We say  $E \in \text{Coh}(X)$  is  $\mu_{\omega,B}$ -(semi)stable, if for any  $0 \neq F \subsetneq E$ ,  $\mu_{\omega,B}(F) < (\leq) \mu_{\omega,B}(E/F)$ . The Harder-Narasimhan (H-N for short) property holds for  $\text{Coh}(X)$  and so we can define the slopes

$$\mu_{\omega,B}^+(E) = \max_{0 \neq G \subseteq E} \mu_{\omega,B}(G), \quad \mu_{\omega,B}^-(E) = \min_{G \subsetneq E} \mu_{\omega,B}(E/G)$$

of  $E \in \text{Coh}(X)$ . Then for a given interval  $I \subset \mathbb{R} \cup \{+\infty\}$ , the subcategory  $\text{HN}_{\omega,B}^\mu(I) \subset \text{Coh}(X)$  is defined by

$$\text{HN}_{\omega,B}^\mu(I) = \{E \in \text{Coh}(X) : [\mu_{\omega,B}^-(E), \mu_{\omega,B}^+(E)] \subset I\}.$$

The subcategories  $\mathcal{T}_{\omega,B}$  and  $\mathcal{F}_{\omega,B}$  of  $\text{Coh}(X)$  are defined by

$$\mathcal{T}_{\omega,B} = \text{HN}_{\omega,B}^\mu(0, +\infty], \quad \mathcal{F}_{\omega,B} = \text{HN}_{\omega,B}^\mu(-\infty, 0].$$

Now  $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$  forms a torsion pair on  $\text{Coh}(X)$  and let the abelian category  $\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle \subset D^b(X)$  be the corresponding tilt of  $\text{Coh}(X)$ . Define the central charge function  $Z_{\omega,B} : K(X) \rightarrow \mathbb{C}$  by  $Z_{\omega,B}(E) = -\int_X e^{-B-i\omega} \text{ch}(E)$ .

Following [BMT], the tilt-slope  $\nu_{\omega,B}(E)$  of  $E \in \mathcal{B}_{\omega,B}$  is defined by

$$\nu_{\omega,B}(E) = \begin{cases} +\infty & \text{if } \omega^2 \text{ch}_1^B(E) = 0 \\ \frac{\Im Z_{\omega,B}(E)}{\omega^2 \text{ch}_1^B(E)} & \text{otherwise.} \end{cases}$$

In [BMT] the notion of  $\nu_{\omega,B}$ -stability for objects in  $\mathcal{B}_{\omega,B}$  is introduced in a similar way to  $\mu_{\omega,B}$ -stability for  $\text{Coh}(X)$ . Also it is proved that the abelian category  $\mathcal{B}_{\omega,B}$  satisfies the H-N

property with respect to  $\nu_{\omega,B}$ -stability. Then one can define the slopes  $\nu_{\omega,B}^+, \nu_{\omega,B}^-$  for objects in  $\mathcal{B}_{\omega,B}$  and the subcategory  $\text{HN}_{\omega,B}^\nu(I) \subset \mathcal{B}_{\omega,B}$  for an interval  $I \subset \mathbb{R} \cup \{+\infty\}$ . The subcategories  $\mathcal{T}'_{\omega,B}$  and  $\mathcal{F}'_{\omega,B}$  of  $\mathcal{B}_{\omega,B}$  are defined by

$$\mathcal{T}'_{\omega,B} = \text{HN}_{\omega,B}^\nu(0, +\infty], \quad \mathcal{F}'_{\omega,B} = \text{HN}_{\omega,B}^\nu(-\infty, 0].$$

Then  $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$  forms a torsion pair on  $\mathcal{B}_{\omega,B}$  and let the abelian category  $\mathcal{A}_{\omega,B} = \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \rangle \subset D^b(X)$  be the corresponding tilt.

**Conjecture 1.1.** ([BMT, Conjecture 3.2.6]) *The pair  $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$  is a Bridgeland stability condition on  $D^b(X)$ .*

**Definition 1.2.** (i) Let  $\mathcal{C}_{\omega,B}$  be the class of  $\nu_{\omega,B}$ -stable objects  $E \in \mathcal{B}_{\omega,B}$  with  $\nu_{\omega,B}(E) = 0$ .  
(ii) Let  $\mathcal{M}_{\omega,B}$  be the class of objects  $E \in \mathcal{C}_{\omega,B}$  with  $\text{Ext}^1(\mathcal{O}_x, E) = 0$  for any  $x \in X$ .

The objects in  $\mathcal{M}_{\omega,B}[1]$  are minimal objects in  $\mathcal{A}_{\omega,B}$  (see [MP, Lemma 2.3]).

Let us assume  $B \in \text{NS}_{\mathbb{Q}}(X)$  and  $\omega \in \text{NS}_{\mathbb{R}}(X)$  is an ample class with  $\omega^2$  is rational. Then similar to the proof of [BMT, Proposition 5.2.2] one can show that the abelian category  $\mathcal{A}_{\omega,B}$  is Noetherian. Therefore Conjecture 1.1 is equivalent to the following (see [BMT, Corollary 5.2.4]).

**Conjecture 1.3.** ([BMT, Conjecture 3.2.7]) *Any  $E \in \mathcal{C}_{\omega,B}$  satisfies the so-called **Bogomolov-Gieseker Type Inequality**:*

$$\Re Z_{\omega,B}(E[1]) < 0, \text{ i.e. } \text{ch}_3^B(E) < \frac{\omega^2}{2} \text{ch}_1^B(E).$$

Moreover in [BMT] they proposed the following strong inequality for objects in  $\mathcal{C}_{\omega,B}$ .

**Conjecture 1.4.** ([BMT, Conjecture 1.3.1]) *Any  $E \in \mathcal{C}_{\omega,B}$  satisfies the so-called **Strong Bogomolov-Gieseker Type Inequality**:*

$$\text{ch}_3^B(E) \leq \frac{\omega^2}{18} \text{ch}_1^B(E).$$

For any  $E \in \mathcal{C}_{\omega,B}$ , there exists  $E' \in \mathcal{M}_{\omega,B}$  such that  $0 \rightarrow E \rightarrow E' \rightarrow T \rightarrow 0$  is a short exact sequence (SES for short) in  $\mathcal{B}_{\omega,B}$  for some  $T \in \text{Coh}^0(X)$  (see [MP, Proposition 2.9]). Therefore one only needs to check the B-G (respectively, strong B-G) type inequality for objects in  $\mathcal{M}_{\omega,B}$ .

**1.2. Autoequivalences of abelian varieties.** First of all we briefly introduce Fourier-Mukai theory (see [BBR, Huy1] for further details). Let  $X, Y$  be smooth projective varieties and let  $p_i, i = 1, 2$  be the projection maps from  $X \times Y$  to  $X$  and  $Y$ , respectively. The Fourier-Mukai functor (FM functor for short)  $\Phi_{\mathcal{E}}^{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y)$  with kernel  $\mathcal{E} \in D^b(X \times Y)$  is defined by

$$\Phi_{\mathcal{E}}^{X \rightarrow Y}(-) = \mathbf{R}p_{2*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)).$$

When  $\Phi_{\mathcal{E}}^{X \rightarrow Y}$  is an equivalence of the derived categories it is called a Fourier-Mukai transform (FMT for short). On the other hand Orlov's representability theorem (see [Orl1]) says that any equivalence between  $D^b(X)$  and  $D^b(Y)$  is isomorphic to  $\Phi_{\mathcal{E}}^{X \rightarrow Y}$  for some  $\mathcal{E} \in D^b(X \times Y)$ . Any FM functor  $\Phi_{\mathcal{E}} : D^b(X) \rightarrow D^b(Y)$  induces a linear map  $\Phi_{\mathcal{E}}^H : H^{2*}(X, \mathbb{Q}) \rightarrow H^{2*}(Y, \mathbb{Q})$  (sometimes called the cohomological FM functor) and it is an isomorphism when  $\Phi_{\mathcal{E}}$  is an FMT.

*Example 1.5.* Let  $(X, L)$  be a principally polarized abelian variety of dimension  $g$  with group operation  $m : X \times X \rightarrow X$ . Then the isogeny  $\phi_L : X \rightarrow \widehat{X}$ ,  $x \mapsto t_x^* L \otimes L^{-1}$  is an isomorphism and also  $\chi(L) = \ell^g/g! = 1$ . In the rest of the paper let  $\Phi : D^b(X) \rightarrow D^b(X)$  be the FMT with the Poincaré line bundle  $\mathcal{P} = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$  on  $X \times X$  as the kernel. In [Muk2] Mukai proved that

- $\Phi$  is an autoequivalence of the derived category  $D^b(X)$ ,
- $\Phi \circ \Phi \cong (-1)^* \text{id}_{D^b(X)}[-g]$ , and
- $(L \circ \Phi)^3 \cong \text{id}_{D^b(X)}[-g]$ .

If we assume the Picard rank of  $X$  is 1 and write  $\text{ch}(E) = (a_0, a_1 \ell, a_2 \ell^2/2!, \dots, a_g \ell^g/g!)$ , then we have  $\text{ch}(\Phi(E)) = \Phi^H(\text{ch}(E)) = (a_g, -a_{g-1} \ell, a_{g-2} \ell^2/2!, \dots, (-1)^g a_0 \ell^g/g!)$  (see [Huy1, Lemma 9.23]).

Following the work of Orlov the group  $\text{Aut } D^b(X)$  of FMTs from  $X$  to  $X$  can be described explicitly as follows (see [Orl2] and [Huy1, Chapter 9] for further details). Let  $X, Y$  be two abelian varieties. Then one can write any morphism  $f : X \times \widehat{X} \rightarrow Y \times \widehat{Y}$  as a matrix  $f = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$  for some morphisms  $p : X \rightarrow Y$ ,  $q : \widehat{X} \rightarrow Y$ ,  $r : X \rightarrow \widehat{Y}$  and  $s : \widehat{X} \rightarrow \widehat{Y}$ . These morphisms have duals:  $\widehat{p} : \widehat{Y} \rightarrow \widehat{X}$ ,  $\widehat{q} : \widehat{Y} \rightarrow X$ ,  $\widehat{r} : Y \rightarrow \widehat{X}$  and  $\widehat{s} : Y \rightarrow X$ . We associate a morphism  $\widetilde{f} : Y \times \widehat{Y} \rightarrow X \times \widehat{X}$  to  $f$  by setting  $\widetilde{f} = \begin{pmatrix} \widehat{s} & -\widehat{q} \\ -\widehat{r} & \widehat{p} \end{pmatrix}$ . Then  $f$  is said to be isometric if it is an isomorphism and its inverse  $f^{-1} \cong \widetilde{f}$ . When  $Y = X$ , we denote the group of all isometric automorphisms of  $X \times \widehat{X}$  by  $U(X \times \widehat{X})$ .

Let  $\Phi_{\mathcal{E}}^{X \rightarrow Y}$  be an FMT between two abelian varieties  $X$  and  $Y$  with kernel  $\mathcal{E} \in D^b(X \times Y)$ . Let us define the map  $\mu_X : X \times X \rightarrow X \times X$  by  $\mu_X(x_1, x_2) = (x_1, m(x_1, x_2))$ . Let  $P_X = p_{14}^* \mathcal{O}_{\Delta} \otimes p_{23}^* \mathcal{P}_X$ , where  $p_{ij}$  are the projection maps from  $(X \times \widehat{X}) \times (X \times X)$ ,  $\mathcal{O}_{\Delta}$  is the structure sheaf on the diagonal  $\Delta \subset X \times X$ , and  $\mathcal{P}_X$  is the Poincaré bundle on  $\widehat{X} \times X$ .

Let  $\mathcal{F} \in D^b(X \times Y)$  be an object such that  $\Phi_{\mathcal{F}}^{Y \rightarrow X} \cong (\Phi_{\mathcal{E}}^{X \rightarrow Y})^{-1}$  and let  $\text{Ad}_{\mathcal{E}}$  be the FMT from  $X \times X$  to  $Y \times Y$  with kernel  $\mathcal{F} \boxtimes \mathcal{E}$ . Then it satisfies

$$\Phi_{\text{Ad}_{\mathcal{E}}(\mathcal{G})}^{Y \rightarrow Y} \cong \Phi_{\mathcal{E}}^{X \rightarrow Y} \circ \Phi_{\mathcal{G}}^{X \rightarrow X} \circ (\Phi_{\mathcal{E}}^{X \rightarrow Y})^{-1}$$

for any  $\mathcal{G} \in D^b(X \times X)$  (see [Orl2]). Now define the equivalence  $F_{\mathcal{E}} : D^b(X \times \widehat{X}) \rightarrow D^b(Y \times \widehat{Y})$  by

$$F_{\mathcal{E}} = \left( \Phi_{P_Y}^{(Y \times \widehat{Y}) \rightarrow (Y \times Y)} \right)^{-1} \circ (\mathbf{R}\mu_{Y*})^{-1} \circ \text{Ad}_{\mathcal{E}} \circ \mathbf{R}\mu_{X*} \circ \Phi_{P_X}^{(X \times \widehat{X}) \rightarrow (X \times X)},$$

so that  $F_{\mathcal{E}}$  fits into the following commutative diagram (see [Huy1, Orl2]).

$$\begin{array}{ccc} D^b(X \times \widehat{X}) & \xrightarrow{F_{\mathcal{E}}} & D^b(Y \times \widehat{Y}) \\ \Phi_{P_X}^{(X \times \widehat{X}) \rightarrow (X \times X)} \downarrow & & \downarrow \Phi_{P_Y}^{(Y \times \widehat{Y}) \rightarrow (Y \times Y)} \\ D^b(X \times X) & & D^b(Y \times Y) \\ \mathbf{R}\mu_{X*} \downarrow & & \downarrow \mathbf{R}\mu_{Y*} \\ D^b(X \times X) & \xrightarrow{\text{Ad}_{\mathcal{E}}} & D^b(Y \times Y) \end{array}$$

The equivalence  $F_{\mathcal{E}}$  can also be expressed in a simple form as follows.

**Lemma 1.6.** ([Huy1, Proposition 9.39], [Orl2]) *The equivalence  $F_{\mathcal{E}}$  is isomorphic to  $f_{\mathcal{E}*}(-) \otimes N_{\mathcal{E}}$  for some line bundle  $N_{\mathcal{E}}$  on  $Y \times \hat{Y}$  and isometric isomorphism  $f_{\mathcal{E}} : X \times \hat{X} \rightarrow Y \times \hat{Y}$ . Moreover,  $f_{\mathcal{E}}(s, \hat{s}) = (t, \hat{t})$  if and only if  $\Phi_{(t, \hat{t})} \circ \Phi_{\mathcal{E}}^{X \rightarrow Y} \cong \Phi_{\mathcal{E}}^{X \rightarrow Y} \circ \Phi_{(s, \hat{s})}$ . Here  $\Phi_{(z, \hat{z})} = t_{z*}(-) \otimes \mathcal{P}_{\hat{z}}$  and  $\mathcal{P}_{\hat{z}}$  is the restriction of the Poincaré line bundle on the product  $Z \times \hat{Z}$ .*

*Example 1.7.* Let  $(X, L)$  be a principally polarized abelian variety. The following examples are important in this paper (see [Huy1, Examples 9.38]). Here  $\delta : X \rightarrow X \times X$  is the diagonal embedding.

$\Phi_{\mathcal{E}}$	$f_{\mathcal{E}}$	$N_{\mathcal{E}}$
$[1] = \Phi_{\mathcal{O}_{\Delta}[1]}^{X \rightarrow X}$	$f_{[1]} = \text{id}_{X \times \hat{X}}$	$\mathcal{O}_{X \times \hat{X}}$
$\Phi_{(s, \hat{s})} = t_{s*}(-) \otimes \mathcal{P}_{\hat{s}}$	$f_{(s, \hat{s})} = \text{id}_{X \times \hat{X}}$	$\mathcal{P}_{\hat{s}} \boxtimes \mathcal{P}_s^*$
$\Phi = \Phi_{\mathcal{P}}^{X \rightarrow X}$	$f_{\mathcal{P}} = \begin{pmatrix} 0 & -\phi_L^{-1} \\ \phi_L & 0 \end{pmatrix}$	$\mathcal{P}_X$
$(-) \otimes L = \Phi_{\delta_* L}^{X \rightarrow X}$	$f_{\delta_* L} = \begin{pmatrix} 1 & 0 \\ -\phi_L & 1 \end{pmatrix}$	$L \boxtimes \mathcal{O}_{\hat{X}}$

Let  $X, Y, Z$  be abelian varieties and let  $\Phi_{\mathcal{E}}^{X \rightarrow Y}, \Phi_{\mathcal{F}}^{Y \rightarrow Z}, \Phi_{\mathcal{G}}^{X \rightarrow Z}$  be FMTs such that  $\Phi_{\mathcal{G}}^{X \rightarrow Z} \cong \Phi_{\mathcal{F}}^{Y \rightarrow Z} \circ \Phi_{\mathcal{E}}^{X \rightarrow Y}$ . Then one can show that  $f_{\mathcal{G}} \cong f_{\mathcal{F}} \circ f_{\mathcal{E}}$  and  $N_{\mathcal{G}} \cong N_{\mathcal{F}} \otimes f_{\mathcal{F}*} N_{\mathcal{E}}$ . So we have a well defined group homomorphism

$$\sigma_X : \text{Aut } D^b(X) \rightarrow U(X \times \hat{X}), \quad \Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}.$$

**Lemma 1.8.** ([Huy1, Proposition 9.55]) *The map  $\sigma_X$  is an epimorphism and its kernel consists of autoequivalences  $\Phi_{(s, \hat{s})}[k]$  where  $s \in X, \hat{s} \in \hat{X}$  and  $k \in \mathbb{Z}$ . So  $\ker \sigma_X \cong \mathbb{Z} \oplus (X \times \hat{X})$ .*

**Notation 1.9.** Assume  $(X, L)$  is a principally polarized abelian variety. Let  $\widetilde{\text{SL}(2, \mathbb{Z})}$  be the central  $\mathbb{Z}$ -extension of the group  $\text{SL}(2, \mathbb{Z})$  generated by the FMTs  $\Phi, (-) \otimes L$  and  $[1]$  as a subgroup of  $\text{Aut } D^b(X)$ . So  $(X \times \hat{X}) \rtimes \widetilde{\text{SL}(2, \mathbb{Z})}$  is a subgroup of  $\text{Aut } D^b(X)$ . Consequently, we have the following diagram (see [Huy1, Chapter 9]):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \oplus (X \times \hat{X}) & \longrightarrow & \text{Aut } D^b(X) & \longrightarrow & U(X \times \hat{X}) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{Z} \oplus (X \times \hat{X}) & \longrightarrow & (X \times \hat{X}) \rtimes \widetilde{\text{SL}(2, \mathbb{Z})} & \longrightarrow & \text{SL}(2, \mathbb{Z}) \longrightarrow 1. \end{array}$$

The isometric automorphism of any FMT in  $(X \times \hat{X}) \rtimes \widetilde{\text{SL}(2, \mathbb{Z})}$  is of the form

$$\begin{pmatrix} x & y\phi_L^{-1} \\ z\phi_L & w \end{pmatrix}$$

for some  $x, y, z, w \in \mathbb{Z}$  satisfying  $xw - yz = 1$ . In the rest of the paper we abuse notation by dropping  $\phi_L, \phi_L^{-1}$  from this matrix. In this way we canonically identify such isometric automorphisms with elements of  $\text{SL}(2, \mathbb{Z})$ .

2. MATRIX REPRESENTATIONS OF  $\mathrm{GL}(2, \mathbb{R})$  AND INDUCED FMTs ON  $H^{2*}(X, \mathbb{Q})$ 

**2.1. Finite dimensional matrix representations of  $\mathrm{GL}(2, \mathbb{R})$ .** Following [Kna], we explicitly construct a variant of the symmetric power representation of all dimensions ( $\geq 2$ ) of  $\mathrm{GL}(2, \mathbb{R})$ .

For  $k \geq 2$ , let  $V_k$  be the vector space of homogeneous polynomials over  $\mathbb{R}$  in variables  $u_1, u_2$  of degree  $k$ . Then  $V_k = \bigoplus_{r=0}^k \mathbb{R} \left( u_1^{k-r} u_2^r \right)$ . So the set

$$\Omega = \left\{ u_1^k, -\binom{k}{1} u_1^{k-1} u_2, \dots, (-1)^r \binom{k}{r} u_1^{k-r} u_2^r, \dots, (-1)^k u_2^k \right\}$$

is a basis of  $V_k$ . Here  $\binom{k}{r} = \begin{cases} \frac{k!}{r!(k-r)!} & \text{if } 0 \leq r \leq k, \\ 0 & \text{otherwise.} \end{cases}$

We have  $\dim_{\mathbb{R}} V_k = k + 1$ . Let us define the map  $\rho^{(k)} : \mathrm{GL}(2, \mathbb{R}) \rightarrow \mathrm{GL}(V_k)$  by

$$\rho^{(k)}(X) \left( Q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = Q \left( X^T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right),$$

for  $X \in \mathrm{GL}(2, \mathbb{R})$  and  $Q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in V_k$ . Then one can easily check that  $\rho^{(k)}$  is a  $(k + 1)$ -dimensional linear representation of  $\mathrm{GL}(2, \mathbb{R})$ . We now explicitly compute the matrix representation of  $\rho^{(k)}$  with respect to the basis  $\Omega$ . Let  $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$  and let  $a_{m,n}^{(k)}(x, y, z, w)$  be the  $(m, n)$ -entry of  $\rho^{(k)}(X)$ . By definition

$$\begin{aligned} (-1)^{n-1} \binom{k}{n-1} (xu_1 + zu_2)^{k-n+1} (yu_1 + wu_2)^{n-1} \\ = \dots + a_{m,n}^{(k)} (-1)^{m-1} \binom{k}{m-1} u_1^{k-m+1} u_2^{m-1} + \dots \end{aligned}$$

By setting  $\lambda = k - m - i + 2$ , we have the following.

**Proposition 2.1.** *The  $(m, n)$ -entry  $a_{m,n}^{(k)}(x, y, z, w)$  of  $\rho^{(k)} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$  is*

$$(-1)^{n-m} \sum_{\lambda \in \mathbb{Z}} \binom{k-m+1}{\lambda-1} \binom{m-1}{n-\lambda} x^{k-m-\lambda+2} y^{\lambda-1} z^{m-n+\lambda-1} w^{n-\lambda}.$$

Here  $a_{m,n}^{(k)}(x, y, z, w)$  are polynomials of  $x, y, z, w$  with coefficients from  $\mathbb{Z}$ . Therefore  $\rho^{(k)}(\mathrm{SL}(2, \mathbb{Z})) \subset \mathrm{GL}(k+1, \mathbb{Z})$ .

**2.2. Induced cohomological FMTs.** We now recall some important notions from finite continued fraction theory (see [HW] for further details). Let  $\mathbf{m} = (m_1, \dots, m_n)$  be a sequence of integers. Define  $s_i, t_i$  for  $0 \leq i \leq n$  by

$$\begin{aligned} s_0 &= 1, \quad s_1 = m_1, \quad s_k = m_k s_{k-1} + s_{k-2} \quad (2 \leq k \leq n), \\ t_0 &= 0, \quad t_1 = 1, \quad t_k = m_k t_{k-1} + t_{k-2} \quad (2 \leq k \leq n). \end{aligned}$$

The key result for us is the following standard fact which we reproduce for the reader's convenience:



**Proposition 2.2.** *If we write the finite continued fraction by*

$$[m_1, m_2, \dots, m_n] = m_1 + \frac{1}{m_2 + \frac{1}{\dots + \frac{1}{m_n}}}$$

*then  $\frac{s_n}{s_{n-1}} = [m_n, \dots, m_1]$ ,  $\frac{t_n}{t_{n-1}} = [m_n, \dots, m_2]$ ,  $\frac{s_n}{t_n} = [m_1, \dots, m_n]$  and  $s_n t_{n-1} - s_{n-1} t_n = (-1)^n$ .*

Let  $(X, L)$  be a  $g$ -dimensional principally polarized abelian variety. The transform  $\Phi_{\mathbf{m}} : D^b(X) \rightarrow D^b(X)$  is defined by

$$\Phi_{\mathbf{m}} := \Phi \circ L^{(-1)^{n+1}m_n} \circ \Phi \circ \dots \circ L^{-m_2} \circ \Phi \circ L^{m_1} \circ \Phi.$$

Here  $\Phi$  is the FMT from  $X$  to  $X$  with the Poincaré line bundle  $\mathcal{P}$  on  $X \times X$  as its kernel and  $L^k$  is  $(-) \otimes L^k$ .

**Proposition 2.3.** *The isometric automorphism associated to the FMT  $\Phi_{\mathbf{m}}$  is*

$$f_{\mathbf{m}} = (-1)^{\frac{n(n+1)}{2}} \begin{pmatrix} (-1)^{n+1}t_n & (-1)^{n+1}s_n \\ t_{n-1} & s_{n-1} \end{pmatrix}.$$

*Proof.* By induction on  $n$ . □

Assume the Picard rank of  $X$  is one and let  $\ell$  be  $c_1(L)$ . As usual, we write  $\text{ch}(E) = (a_0, a_1, a_2, \dots, a_g)$  and so the induced transform on  $H^{2*}(X, \mathbb{Q})$  can be expressed as a  $(g+1) \times (g+1)$  invertible matrix.

*Example 2.4.* The following examples of induced FMTs on  $H^{2*}(X, \mathbb{Q})$  are important in this paper. We identify them in matrix form as images of the corresponding isometric automorphisms under  $\rho^{(g)}$  as given by Proposition 2.1.

$\Phi_{\mathcal{E}}$	$\Phi_{\mathcal{E}}^H$
$[1] = \Phi_{\mathcal{O}_{\Delta}[1]}^{X \rightarrow X}$	$-I_g = -\rho^{(g)}(f_{[1]})$
$\Phi_{(x, \hat{x})} = t_{x*}(-) \otimes \mathcal{P}_{\hat{x}}$	$I_g = \rho^{(g)}(f_{(x, \hat{x})})$
$\Phi = \Phi_{\mathcal{P}}^{X \rightarrow X}$	$\text{adiag}(1, -1, \dots, (-1)^g) = \rho^{(g)}(f_{\mathcal{P}})$
$(-) \otimes L = \Phi_{\delta_* L}^{X \rightarrow X}$	$\binom{i-1}{j-1}_{1 \leq i, j \leq g+1} = \rho^{(g)}(f_{\delta_* L})$

Since  $(X, L)$  is principally polarized, any FMT  $\Phi_{\mathcal{E}}$  in  $(X \times \widehat{X}) \rtimes \widetilde{\text{SL}(2, \mathbb{Z})}$  is isomorphic to  $\Phi_{\mathbf{m}} \circ \Phi_{(s, \hat{s})} \circ [p]$  for some sequence of integers  $\mathbf{m}$ ,  $s \in X$ ,  $\hat{s} \in \widehat{X}$  and  $p \in \mathbb{Z}$ . The induced transform  $\Phi_{\mathcal{E}}^H$  on  $H^{2*}(X, \mathbb{Q})$  gives a well defined group homomorphism

$$(X \times \widehat{X}) \rtimes \widetilde{\text{SL}(2, \mathbb{Z})} \rightarrow \text{GL}(H^{2*}(X, \mathbb{Q})), \quad \Phi_{\mathcal{E}} \mapsto \Phi_{\mathcal{E}}^H$$

given by  $\Phi_{\mathcal{E}}^H = (-1)^p \Phi_{\mathbf{m}}^H$  and  $\Phi_{\mathbf{m}}^H = \rho^{(g)}(f_{\mathbf{m}})$ . Also  $f_{\mathcal{E}} = f_{\mathbf{m}}$ .

For  $\mathbf{m} = (m_1, \dots, m_n)$ , let  $x = (-1)^{\frac{(n+1)(n+2)}{2}} t_n$ ,  $y = (-1)^{\frac{(n+1)(n+2)}{2}} s_n$ ,  $z = (-1)^{\frac{n(n+1)}{2}} t_{n-1}$  and  $w = (-1)^{\frac{n(n+1)}{2}} s_{n-1}$ . By Proposition 2.3, the induced transform on  $H^{2*}(X, \mathbb{Q})$  is  $\Phi_{\mathbf{m}}^H = \rho^{(g)} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$  and its  $(m, n)$ -entry is given explicitly in Proposition 2.1.

Up to shift, any non-trivial FMT in  $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$  is isomorphic to some FMT  $\Phi_{\mathcal{E}}$  with a universal bundle  $\mathcal{E}$  on  $X \times X$  as the kernel. Therefore  $\Phi_{\mathcal{E}}^H = \rho^{(g)} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$  for some  $x, y, z, w \in \mathbb{Z}$  such that  $xw - yz = 1$  and  $\mathrm{rk}(\mathcal{E}_{\{s\} \times X}) = \mathrm{ch}_0(\Phi_{\mathcal{E}}(\mathcal{O}_s)) = (-1)^g y^g > 0$  for any  $s \in X$ .

*Example 2.5.* For the case  $g = 2$

$$\Phi_{\mathcal{E}}^H = \begin{pmatrix} x^2 & -2xy & y^2 \\ -xz & xw + yz & -yw \\ z^2 & -2zw & w^2 \end{pmatrix}.$$

*Example 2.6.* For the case  $g = 3$

$$\Phi_{\mathcal{E}}^H = \begin{pmatrix} x^3 & -3x^2y & 3xy^2 & -y^3 \\ -x^2z & x^2w + 2xyz & -y^2z - 2xyw & y^2w \\ xz^2 & -yz^2 - 2xzw & xw^2 + 2yzw & -yw^2 \\ -z^3 & 3z^2w & -3zw^2 & w^3 \end{pmatrix}.$$

Let us simply denote the twisted Chern character  $\mathrm{ch}^{b\ell}$  by  $\mathrm{ch}^b$ . Since  $((-) \otimes L^k)^H = \rho^{(g)} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$ , we have

$$\begin{aligned} \mathrm{ch}^{-w/y}(\Phi_{\mathcal{E}}(E)) &= e^{w\ell/y} \Phi_{\mathcal{E}}^H \left( e^{x\ell/y} \mathrm{ch}^{x/y}(E) \right) \\ &= \rho^{(g)} \begin{pmatrix} 1 & 0 \\ -w/y & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x/y & 1 \end{pmatrix} \mathrm{ch}^{x/y}(E). \end{aligned}$$

Since  $xw - yz = 1$ , we obtain the following presentation.

**Theorem 2.7.**

$$\begin{aligned} \mathrm{ch}^{-w/y}(\Phi_{\mathcal{E}}(E)) &= \rho^{(g)} \begin{pmatrix} 0 & y \\ -1/y & 0 \end{pmatrix} \mathrm{ch}^{x/y}(E) \\ &= (-1)^g y^g \mathrm{diag} \left( 1, \frac{-1}{y^2}, \dots, \frac{(-1)^{g-1}}{y^{2(g-1)}}, \frac{(-1)^g}{y^{2g}} \right) \mathrm{ch}^{x/y}(E). \end{aligned}$$

*Remark 2.8.* As a result of this theorem, we can see that the induced transform on  $H^{2*}(X, \mathbb{Q})$  of any non-trivial FMT in  $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$  with respect to the appropriate twisted Chern characters looks somewhat similar to the induced transform of  $\Phi$  on  $H^{2*}(X, \mathbb{Q})$  with usual Chern characters.

### 3. ACTION OF FMTs ON STABILITY CONDITIONS

A Bridgeland stability condition  $\sigma$  on a triangulated category  $\mathcal{D}$  consists of a stability function  $Z$  together with a slicing  $\mathcal{P}$  of  $\mathcal{D}$  satisfying certain axioms. Equivalently, one can define  $\sigma$  by giving a bounded t-structure on  $\mathcal{D}$  together with a stability function  $Z$  on the corresponding heart  $\mathcal{A}$  satisfying the H-N property. Then  $\sigma$  is usually written as the pair

$(Z, \mathcal{P})$  or  $(Z, \mathcal{A})$ . See [Bri1], [Huy3] or [BBR, Appendix D] for further details. Let  $\Upsilon \in \text{Aut } \mathcal{D}$  and let  $W : K(\mathcal{D}) \rightarrow \mathbb{C}$  be a group homomorphism. Then

$$(\Upsilon \cdot W)([E]) = W([\Upsilon^{-1}(E)])$$

defines a left action of the group  $\text{Aut } \mathcal{D}$  on  $\text{Hom}(K(\mathcal{D}), \mathbb{C})$ . Moreover this can be extended to the natural left action of  $\text{Aut } \mathcal{D}$  on the space of all stability conditions on  $\mathcal{D}$  by defining  $\Upsilon \cdot (Z, \mathcal{A}) = (\Upsilon \cdot Z, \Upsilon(\mathcal{A}))$ .

Let  $(X, L)$  be a principally polarized  $g$ -dimensional abelian variety with Picard rank one and let  $\ell$  be  $c_1(L)$ . Then the Todd class of  $X$  is trivial and so for any object in  $D^b(X)$  the Mukai vector is the Chern character. Any complexified class in  $\text{NS}_{\mathbb{C}}(X)$  is of the form  $u\ell$  for some  $u = b + im \in \mathbb{C}$ , where  $b, m \in \mathbb{R}$ . Assume  $m \neq 0$ . Define the function  $Z_{u\ell} : K(X) \rightarrow \mathbb{C}$  by

$$Z_{u\ell}(E) = - \int_X e^{-u\ell} \text{ch}(E).$$

If we denote the Mukai pairing on  $X$  by  $\langle -, - \rangle$  then  $Z_{u\ell}(E) = \langle e^{u\ell}, \text{ch}(E) \rangle$ . It is expected that  $Z_{u\ell}$  is a central charge of some stability condition on  $X$  (see [BMT, Conjecture 2.1.2], [Pol]). This is already known to be true for  $g = 1, 2$  completely, and the authors proved the case of  $m = \sqrt{3}/2$ ,  $b = 1/2$  for  $g = 3$  in [MP, Theorem 3.3].

Let  $\Phi_{\mathcal{E}}$  be a non-trivial FMT in  $(X \times \widehat{X}) \rtimes \widetilde{\text{SL}(2, \mathbb{Z})}$  with kernel the universal bundle  $\mathcal{E}$  on  $X \times X$ . From the previous section, the induced transform on  $H^{2*}(X, \mathbb{Q})$  is  $\Phi_{\mathcal{E}}^H = \rho^{(g)} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$  for some  $x, y, z, w \in \mathbb{Z}$  satisfying  $xw - yz = 1$  and  $(-1)^g y^g > 0$ . Also  $e^{k\ell} = ((-) \otimes L^k)^H(\text{ch}(\mathcal{O})) = \rho^{(g)} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \text{ch}(\mathcal{O})$ . Then

$$\Phi_{\mathcal{E}}^H(e^{u\ell}) = \rho^{(g)} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \text{ch}(\mathcal{O}) = \rho^{(g)} \begin{pmatrix} x - yu & y \\ z - wu & w \end{pmatrix} \text{ch}(\mathcal{O})$$

and from Proposition 2.1 it is equal to  $(x - yu)^g e^{(-z + wu)\ell/(x - yu)}$ .

By Căldăraru-Willertons' generalization, the cohomological FMTs are isometries with respect to the Mukai pairing (see [CW], [Huy1, Proposition 5.44]). Therefore for any  $E \in D^b(X)$  we have

$$(\Phi_{\mathcal{E}} \cdot Z_{u\ell})(E) = \left\langle e^{u\ell}, \text{ch}(\Phi_{\mathcal{E}}^{-1}(E)) \right\rangle = \left\langle \Phi_{\mathcal{E}}^H(e^{u\ell}), \text{ch}(E) \right\rangle.$$

So the function  $Z_{u\ell} \in \text{Hom}(K(X), \mathbb{C})$  satisfies the following key relation under the action of  $\text{Aut } D^b(X)$ .

**Proposition 3.1.** *We have  $\Phi_{\mathcal{E}} \cdot Z_{u\ell} = \zeta Z_{v\ell}$  for  $\zeta = (x - yu)^g$  and  $v = (-z + wu)/(x - yu)$ .*

**Note 3.2.** Assume there exist a stability condition for any complexified ample class  $\theta\ell$  with a heart  $\mathcal{A}_{\theta\ell}$  and slicing  $\mathcal{P}_{\theta\ell}$  associated to the central charge function  $Z_{\theta\ell}$ . From Proposition 3.1 for any  $\phi \in \mathbb{R}$ ,  $\zeta Z_{v\ell}(\Phi_{\mathcal{E}}(\mathcal{P}_{u\ell}(\phi))) \subset \mathbb{R}_{>0} e^{i\pi\phi}$ ; that is

$$Z_{v\ell}(\Phi_{\mathcal{E}}(\mathcal{P}_{u\ell}(\phi))) \subset \mathbb{R}_{>0} e^{i(\pi\phi - \arg(\zeta))}.$$

So we would expect

$$\Phi_{\mathcal{E}}(\mathcal{P}_{u\ell}(\phi)) = \mathcal{P}_{v\ell} \left( \phi - \frac{\arg(\zeta)}{\pi} \right),$$

and so

$$\Phi_{\mathcal{E}}(\mathcal{P}_{v\ell}(0, 1]) = \mathcal{P}_{v\ell}\left(-\frac{\arg(\zeta)}{\pi}, -\frac{\arg(\zeta)}{\pi} + 1\right].$$

For  $0 \leq \alpha < 1$ ,  $\mathcal{P}_{v\ell}(\alpha, \alpha + 1] = \langle \mathcal{P}_{v\ell}(0, \alpha] [1], \mathcal{P}_{v\ell}(\alpha, 1] \rangle$  is a tilt of  $\mathcal{A}_{v\ell} = \mathcal{P}_{v\ell}(0, 1]$  associated to a torsion theory coming from  $Z_{v\ell}$ -stability. Therefore, one would expect  $\Phi_{\mathcal{E}}(\mathcal{A}_{v\ell})$  is a tilt of  $\mathcal{A}_{v\ell}$  associated to a torsion theory coming from  $Z_{v\ell}$ -stability, up to shift.

Moreover, for the nontrivial FMT  $\Phi_{\mathcal{E}}$  (i.e.  $y \neq 0$ ), when  $\zeta = (x - yu)^g$  is real, we would expect the equivalence (for some integer  $q$ )

$$\Phi_{\mathcal{E}}(\mathcal{A}_{v\ell}) \cong \mathcal{A}_{v\ell}[q].$$

Here  $u = x/y + \lambda e^{il\pi/g}$  and  $v = -w/y - \frac{1}{\lambda y^2} e^{-il\pi/g}$  for some  $l \in \mathbb{Z} \setminus g\mathbb{Z}$  and  $0 \neq \lambda \in \mathbb{R}$ . The numerology given for  $g = 3$  case is very important in the rest of this paper.

*Remark 3.3.* Let us consider the numerology for  $g = 2$  case. For  $b, m \in \mathbb{Q}$  with  $m > 0$ , recall  $\mathcal{B}_{m\ell, b\ell}$  is the tilt of  $\text{Coh}(X)$  with respect to the torsion theory coming from twisted slope  $\mu_{m\ell, b\ell}$ -stability on  $\text{Coh}(X)$ . Since this category is independent of  $m$ , we simply denote it by  $\mathcal{B}_b$ . Following the ideas in [Yos] together with the representation of the induced transform on  $H^{2*}(X, \mathbb{Q})$  with respect to the twisted Chern characters as in Theorem 2.7, one can prove the equivalence

$$\Phi_{\mathcal{E}}[1](\mathcal{B}_b) \cong \mathcal{B}_{b'},$$

where  $b = x/y$  and  $b' = -w/y$  (see [Huy2]).

#### 4. RELATION OF FMTs TO THE STRONG B-G TYPE INEQUALITY

**4.1. Some properties of  $\mathcal{A}_{\omega, B}$ .** Let  $(X, L)$  be a polarized projective threefold with Picard rank one and let  $\ell$  be  $c_1(L)$ . Let  $D, B$  be in  $\text{NS}_{\mathbb{Q}}(X)$ . Then there exists  $b \in \mathbb{Q}$  such that  $B = b\ell$ . Assume  $b > 0$ . Then with respect to the twisted Chern character  $\text{ch}^D$  the central charge function is

$$Z_{\sqrt{3}B, D+B}(-) = - \int_X e^{-(B+i\sqrt{3}B)} \text{ch}^D(-).$$

So for  $E \in D^b(X)$ ,  $\Im Z_{\sqrt{3}B, D+B}(E) = \sqrt{3}b\ell (\text{ch}_2^D(E) - b\ell \text{ch}_1^D(E))$ .

**Proposition 4.1.** *Let  $E \in \mathcal{B}_{\sqrt{3}B, D+B}$  and let  $E_i = H_{\text{Coh}(X)}^i(E)$ . Let  $E_i^{\pm}$  be the  $H$ - $N$  semistable factors of  $E_i$  with highest and lowest  $\mu_{\sqrt{3}B, D+B}$  slopes. Then we have the following:*

- (i) if  $E \in \text{HN}_{\sqrt{3}B, D+B}^{\nu}(-\infty, 0)$  and  $E_{-1} \neq 0$ , then  $\ell^2 \text{ch}_1^D(E_{-1}^+) < 0$ ;
- (ii) if  $E \in \text{HN}_{\sqrt{3}B, D+B}^{\nu}(0, +\infty]$  and  $\text{rk}(E_0) \neq 0$ , then  $\ell^2 \text{ch}_1^D(E_0^-) > 2b\ell^3 \text{ch}_0^D(E_0^-)$ ; and
- (iii) if  $E$  is tilt-stable with  $\nu_{\sqrt{3}B, D+B}(E) = 0$ , then
  - (a) for  $E_{-1} \neq 0$ ,  $\ell^2 \text{ch}_1^D(E_{-1}) \leq 0$  with equality if and only if  $\text{ch}_2^D(E_{-1}) = 0$ , and
  - (b) for  $\text{rk}(E_0) \neq 0$ ,  $\ell^2 \text{ch}_1^D(E_0) \geq 2b\ell^3 \text{ch}_0^D(E_0)$  with equality if and only if  $\text{ch}_2^D(E_0) = 2b^2\ell^2 \text{ch}_0^D(E_0)$ .

*Proof.* For a slope semistable torsion free sheaf  $G$ , the usual B-G inequality in terms of the twisted Chern character is  $(\text{ch}_1^D(G))^2 \ell \geq 2 \text{ch}_0^D(G) \text{ch}_2^D(G) \ell$ . The proposition follows in exactly the same way as [MP, Proposition 3.1].  $\square$

We have  $Z_{\sqrt{3}B, D-B}(-) = - \int_X e^{-(B+i\sqrt{3}B)} \text{ch}^{D-2B}(-)$  and also  $\text{ch}^{D-2B} = e^{2B} \text{ch}^D$ . Therefore from the above proposition we get the following.

**Proposition 4.2.** *Let  $E \in \mathcal{B}_{\sqrt{3}B, D-B}$  and let  $E_i = H_{\text{Coh}(X)}^i(E)$ . Let  $E_i^\pm$  be the H-N semistable factors of  $E_i$  with highest and lowest  $\mu_{\sqrt{3}B, D-B}$  slopes. Then we have*

- (i) *if  $E \in \text{HN}_{\sqrt{3}B, D-B}^\nu(-\infty, 0)$  and  $E_{-1} \neq 0$ , then  $\ell^2 \text{ch}_1^D(E_{-1}^+) < -2b\ell^3 \text{ch}_0^D(E_{-1}^+)$ ;*
- (ii) *if  $E \in \text{HN}_{\sqrt{3}B, D-B}^\nu(0, +\infty]$  and  $\text{rk}(E_0) \neq 0$ , then  $\ell^2 \text{ch}_1^D(E_0^-) > 0$ ; and*
- (iii) *if  $E$  is tilt-stable with  $\nu_{\sqrt{3}B, D-B}(E) = 0$ , then*
  - (a) *for  $E_{-1} \neq 0$ ,  $\ell^2 \text{ch}_1^D(E_{-1}) \leq -2b\ell^3 \text{ch}_0^D(E_{-1})$  with equality if and only if  $\text{ch}_2^D(E_{-1}) = 2b^2\ell^2 \text{ch}_0^D(E_{-1})$ , and*
  - (b) *for  $\text{rk}(E_0) \neq 0$ ,  $\ell^2 \text{ch}_1^D(E_0) \geq 0$  with equality if and only if  $\text{ch}_2^D(E_0) = 0$ .*

**4.2. Relation of FMTs to stability conditions.** From here onwards let  $(X, L)$  be a principally polarized abelian threefold with Picard rank one and let  $\ell$  be  $c_1(L)$ . As before, we also abbreviate the twisted Chern character  $\text{ch}^{b\ell}(E) = e^{-b\ell} \text{ch}(E)$  by  $\text{ch}^b(E)$ .

Similar to [MP, Example 2.5] we can identify some examples of minimal objects of any  $\mathcal{A}_{\omega, B}$  as follows.

*Example 4.3.* Let  $p, q \in \mathbb{Q}$  and  $q > 0$ . There exist simple semi-homogeneous vector bundles  $\mathcal{E}_s^\pm$  parameterized by  $s \in X$  having the Chern character  $(u^3, u^2v, uv^2, v^3)$  with  $u, v \in \mathbb{Z}$  such that  $u > 0$ ,  $\gcd(u, v) = 1$  and  $v/u = p \pm q$ . Here the vector bundles  $\mathcal{E}_s^\pm$  are restrictions of the universal bundles  $\mathcal{E}^\pm$  on  $X \times X$  associated to FMTs. Also  $\mathcal{E}_s^\pm$  are slope stable (see [Muk1, Proposition 6.16]). Then the discriminant in the sense of Drežet  $\overline{\Delta}_{\sqrt{3}q\ell}(\mathcal{E}_s^\pm) = 0$  and so by [BMT, Proposition 7.4.1]  $\mathcal{E}_s^+, \mathcal{E}_s^-[1] \in \mathcal{B}_{\sqrt{3}q\ell, p\ell}$  are  $\nu_{\sqrt{3}q\ell, p\ell}$ -stable. Also we have  $\Im Z_{\sqrt{3}q\ell, p\ell}(\mathcal{E}_s^\pm) = 0$  and  $\text{ch}_1^p(\mathcal{E}_s^\pm) \neq 0$  and so  $\nu_{\sqrt{3}q\ell, p\ell}(\mathcal{E}_s^+) = \nu_{\sqrt{3}q\ell, p\ell}(\mathcal{E}_s^-[1]) = 0$ . Therefore by [MP, Lemma 2.3]  $\mathcal{E}_s^+[1], \mathcal{E}_s^-[2] \in \mathcal{A}_{\sqrt{3}q\ell, p\ell}$  are minimal objects. Moreover one can check by direct computation that  $\mathcal{E}_s^+, \mathcal{E}_s^-[1] \in \mathcal{M}_{\sqrt{3}q\ell, p\ell}$  satisfy the strong B-G type inequality.

Let  $\Upsilon$  be a non-trivial FMT in  $(X \times \widehat{X}) \rtimes \widetilde{\text{SL}(2, \mathbb{Z})}$  with kernel the universal bundle  $\mathcal{E}$  on  $X \times X$ . Then the induced transform on  $H^{2*}(X, \mathbb{Q})$  is  $\Upsilon^H = \rho \begin{pmatrix} x & y \\ z & w \end{pmatrix}$  for some  $x, y, z, w \in \mathbb{Z}$  with  $xw - yz = 1$  and  $y < 0$  (see Example 2.6). Here and in the rest of the paper we write  $\rho$  for  $\rho^{(3)}$ . Now we have  $\text{ch}(\mathcal{E}_{\{s\} \times X}) = (-y^3, y^2w, -yw^2, w^3)$ . Let  $\widehat{\Upsilon}$  be the FMT with kernel given by  $\Sigma^* \mathcal{E}^*$ , where

$$\Sigma : X \times X \rightarrow X \times X : (x_1, x_2) \mapsto (x_2, x_1)$$

switches the factors. Then  $\widehat{\Upsilon}[3]$  is the quasi-inverse of  $\Upsilon$  and so we have  $\widehat{\Upsilon}^H = \rho \begin{pmatrix} -w & y \\ z & -x \end{pmatrix}$ .

Also  $\text{ch}(\mathcal{E}_{X \times \{s\}}^*) = (-y^3, -y^2x, -yx^2, -x^3)$ . For  $g = 3$  case, Theorem 2.7 says

$$\text{ch}^{-w/y}(\Upsilon(E)) = \rho \begin{pmatrix} 0 & y \\ -1/y & 0 \end{pmatrix} \text{ch}^{x/y}(E) = \text{adiag} \left( -y^3, y, -\frac{1}{y}, \frac{1}{y^3} \right) \text{ch}^{x/y}(E),$$

and we have  $\text{ch}^{-w/y}(\mathcal{E}_{\{s\} \times X}) = (-y^3, 0, 0, 0)$  and  $\text{ch}^{x/y}(\mathcal{E}_{X \times \{s\}}^*) = (-y^3, 0, 0, 0)$ .

For some given  $\lambda \in \mathbb{Q}_{>0}$ , let

$$b = \left( \frac{x}{y} + \frac{\lambda}{2} \right), \quad m = \frac{\sqrt{3}\lambda}{2}, \quad b' = \left( -\frac{w}{y} - \frac{1}{2\lambda y^2} \right) \quad \text{and} \quad m' = \frac{\sqrt{3}}{2\lambda y^2}.$$

**Proposition 4.4.** *For  $E \in D^b(X)$ , we have*

- if  $\text{ch}^{x/y}(E) = (a_0, a_1, a_2, a_3)$  then  $\Im Z_{m\ell, b\ell}(E) = \frac{3\sqrt{3}\lambda}{2}(a_2 - \lambda a_1)$ , and
- if  $\text{ch}^{-w/y}(E) = (a_0, a_1, a_2, a_3)$  then  $\Im Z_{m'\ell, b'\ell}(E) = \frac{3\sqrt{3}}{2\lambda y^2} \left( a_2 + \frac{1}{\lambda y^2} a_1 \right)$ .

The following result is a generalization of [MP, Proposition 6.2].

**Proposition 4.5.** *For  $E \in D^b(X)$ , we have*

$$\begin{aligned} \Im Z_{m'\ell, b'\ell}(\Upsilon(E)) &= -\frac{1}{|\lambda y|^3} \Im Z_{m\ell, b\ell}(E), \text{ and} \\ \Im Z_{m\ell, b\ell}(\hat{\Upsilon}[1](E)) &= -|\lambda y|^3 \Im Z_{m'\ell, b'\ell}(E). \end{aligned}$$

*Proof.* Let  $\text{ch}^{x/y}(E) = (a_0, a_1, a_2, a_3)$ . Then by Proposition 4.4  $\Im Z_{m\ell, b\ell}(E) = \frac{3\sqrt{3}\lambda}{2}(a_2 - \lambda a_1)$ . By Theorem 2.7 we have  $\text{ch}^{-w/y}(\Upsilon(E)) = (-y^3 a_3, y a_2, -a_1/y, a_0/y^3)$ . So by Proposition 4.4

$$\Im Z_{m'\ell, b'\ell}(\Upsilon(E)) = \frac{3\sqrt{3}}{2\lambda y^2} \left( -\frac{1}{y} a_1 + \frac{1}{\lambda y^2} y a_2 \right) = \frac{1}{\lambda^3 y^3} \Im Z_{m\ell, b\ell}(E).$$

The result follows as  $y < 0$ . Similarly one can prove the other equality.  $\square$

The aim of the next sections is to prove the following equivalences of abelian categories which generalizes [MP, Theorem 6.10].

**Theorem 4.6.** *The FMTs  $\Upsilon[1]$  and  $\hat{\Upsilon}[2]$  give the equivalences of abelian categories*

$$\Upsilon[1](\mathcal{A}_{m\ell, b\ell}) \cong \mathcal{A}_{m'\ell, b'\ell} \text{ and } \hat{\Upsilon}[2](\mathcal{A}_{m'\ell, b'\ell}) \cong \mathcal{A}_{m\ell, b\ell}.$$

*Remark 4.7.* One can see that  $b, m, b', m'$  in the above theorem are exactly the numbers given for  $g = 3$  case in Note 3.2. Moreover, the shifts are compatible with the images of  $\mathcal{O}_x$  under the FMTs that are minimal objects in the corresponding abelian categories, as discussed in Example 4.3.

The notion of tilt stability can be extended from rational to real as considered in [Mac2] for  $\mathbb{P}^3$ . As a result of the above theorem we get the following.

**Theorem 4.8.** *The strong B-G type inequality holds for tilt stable objects of  $X$  with zero tilt slope.*

*Proof.* By [Mac2, Proposition 2.4] it is enough to consider a dense family of classes  $\omega = \alpha\ell$ ,  $B = \beta\ell$  such that  $\alpha/\sqrt{3} \in \mathbb{Q}_{>0}$ ,  $\beta \in \mathbb{Q}$ . Then for given  $\alpha, \beta$  one can easily find  $x, y \in \mathbb{Z}$ ,  $\lambda \in \mathbb{Q}$  such that  $\gcd(x, y) = 1$ ,  $\alpha = \sqrt{3}\lambda/2$ ,  $\beta = x/y + \lambda/2$ . Now using the Euclid algorithm and Proposition 2.2 (for example, see Appendix A of [BH]), one can find a non-trivial FMT  $\Upsilon$  which gives the equivalence of abelian categories as in Theorem 4.6. Therefore we only need to prove the claim for objects in  $\mathcal{C}_{m\ell, b\ell}$ .

By [MP, Proposition 2.9] it is enough to check that the strong B-G type inequality is satisfied by each object in  $\mathcal{M}_{m\ell, b\ell} \subset \mathcal{C}_{m\ell, b\ell}$ . Moreover, the objects in  $\{M : M \cong \mathcal{E}_{X \times \{s\}}^*[1] \text{ for some } s \in X\} \subset \mathcal{M}_{\omega, B}$  satisfy the strong B-G type inequality (see Example 4.3). So we only need to check the strong B-G type inequality for objects in  $\mathcal{M}_{m\ell, b\ell} \setminus \{M : M \cong \mathcal{E}_{X \times \{s\}}^*[1] \text{ for some } s \in X\}$ .

Let  $E \in \mathcal{M}_{m\ell, b\ell} \setminus \{M : M \cong \mathcal{E}_{X \times \{s\}}^*[1] \text{ for some } s \in X\}$ . Then  $E[1] \in \mathcal{A}_{m\ell, b\ell}$  is a minimal object and so by the equivalence in Theorem 4.6  $\Upsilon[1](E[1]) \in \mathcal{A}_{m'\ell, b'\ell}$  is also a minimal object. So  $\Upsilon[1](E[1]) \in \mathcal{F}'_{m'\ell, b'\ell}[1]$  or  $\Upsilon[1](E[1]) \in \mathcal{T}'_{m'\ell, b'\ell}$ . By Proposition 4.5,

$\Im Z_{m'\ell, b'\ell}(\Upsilon[1](E[1])) = 0$ . Suppose  $\Upsilon[1](E[1]) \in \mathcal{T}'_{m'\ell, b'\ell}$ . Since  $\nu_{m'\ell, b'\ell}(\Upsilon[1](E[1])) > 0$  and  $\Im Z_{m'\ell, b'\ell}(\Upsilon[1](E[1])) = 0$ , we have  $m'^2 \ell^2 \text{ch}_1^{b'}(\Upsilon[1](E[1])) = 0$ . By [MP, Lemma 1.1 (iii)]  $\Upsilon[1](E[1]) \in \text{Coh}^0(X)$ , and so  $E \in \widehat{\Upsilon}[1](\text{Coh}^0(X))$ . That is,  $E$  has a filtration of objects from  $\{M : M \cong \mathcal{E}_{X \times \{s\}}^*[1] \text{ for some } s \in X\}$ ; which is not possible. So  $\Upsilon[1](E) \in \mathcal{C}_{m'\ell, b'\ell}$ . Moreover, for any  $x \in X$  we have

$$\text{Ext}^1(\mathcal{O}_s, \Upsilon[1](E)) \cong \text{Hom}(\mathcal{O}_s, \Upsilon[2](E)) \cong \text{Hom}(\mathcal{E}_{X \times \{s\}}^*[1], E) = 0$$

as  $E \not\cong \mathcal{E}_{X \times \{s\}}^*[1]$ . Hence  $\Upsilon[1](E) \in \mathcal{M}_{m'\ell, b'\ell}$ .

Let  $\text{ch}^{x/y}(E) = (a_0, a_1, a_2, a_3)$ . Write  $F = \Upsilon[1](E)$  and let  $F_i = H_{\text{Coh}(X)}^i(F)$ . Then  $\Im Z_{m\ell, b\ell}(E) = 0$  implies  $a_2 = \lambda a_1$  (see Proposition 4.4). Now the strong B-G type inequality says

$$a_3 - \lambda^2 a_1 \leq 0.$$

By Theorem 2.7,  $\text{ch}^{-w/y}(F) = (y^3 a_3, -y \lambda a_1, a_1/y, -a_0/y^3)$ . Then by Proposition 4.2 we have

$$\ell^2 \text{ch}_1^{-w/y}(F_{-1}) \leq -\frac{1}{\lambda y^2} \ell^3 \text{ch}_0^{-w/y}(F_{-1}) \quad \text{and} \quad \ell^2 \text{ch}_1^{-w/y}(F_0) \geq 0.$$

Therefore  $\ell^2 \text{ch}_1^{-w/y}(F) \geq -\frac{1}{\lambda y^2} \ell^3 \text{ch}_0^{-w/y}(F)$ . That is  $-y \lambda a_1 \geq -\frac{1}{\lambda y^2} y^3 a_3$  and so  $\lambda^2 a_1 \geq a_3$  as required.  $\square$

Then we can deduce the main theorem of this paper:

**Theorem 4.9.** *Let  $\alpha, \beta$  such that  $\alpha/\sqrt{3} \in \mathbb{Q}_{>0}$  and  $\beta \in \mathbb{Q}$ . Then the pair  $(\mathcal{A}_{\alpha\ell, \beta\ell}, Z_{\alpha\ell, \beta\ell})$  defines a Bridgeland stability condition on  $D^b(X)$ .*

## 5. FOURIER-MUKAI TRANSFORMS ON $\text{Coh}(X)$ AND $\mathcal{B}_{\omega, B}$

Let us continue the setting introduced in subsection 4.2 for the principally polarized abelian threefold  $(X, L)$  with Picard rank one.

For  $E \in \text{Coh}(X)$  and  $q \in \mathbb{Q}$ , define the twisted slope  $\mu_q(E) = \mu_{\ell/\sqrt{6}, q\ell}(E)$ . If  $\text{ch}(E) = (a_0, a_1, a_2, a_3)$  then  $\mu_q(E) = a_1/a_0 - q$  when  $a_0 \neq 0$ , and  $\mu_q(E) = +\infty$  when  $a_0 = 0$ . In the rest of the paper we mostly use  $\mu_q$  slope for coherent sheaves and we simply write  $\text{HN}_q = \text{HN}_{\ell/\sqrt{6}, q\ell}^\mu$ . Moreover define  $\mathcal{T}_q = \text{HN}_q(0, +\infty]$  and  $\mathcal{F}_q = \text{HN}_q(-\infty, 0]$ .

The isomorphisms  $\widehat{\Upsilon} \circ \Upsilon \cong \text{id}_{D^b(X)}[-3]$  and  $\Upsilon \circ \widehat{\Upsilon} \cong \text{id}_{D^b(X)}[-3]$  give us the following convergence of spectral sequences.

### Mukai Spectral Sequence 5.1.

$$\begin{aligned} E_2^{p,q} &= \widehat{\Upsilon}_{\text{Coh}(X)}^p \Upsilon_{\text{Coh}(X)}^q(E) \implies H_{\text{Coh}(X)}^{p+q-3}(E), \\ E_2^{p,q} &= \Upsilon_{\text{Coh}(X)}^p \widehat{\Upsilon}_{\text{Coh}(X)}^q(E) \implies H_{\text{Coh}(X)}^{p+q-3}(E), \end{aligned}$$

for  $E$ . Here  $\Upsilon_{\text{Coh}(X)}^i(-) = H_{\text{Coh}(X)}^i(\Upsilon(-))$ .

For  $E \in \text{Coh}(X)$ , from the above spectral sequences we immediately have  $\widehat{\Upsilon}_{\text{Coh}(X)}^0 \Upsilon_{\text{Coh}(X)}^0(E) = \widehat{\Upsilon}_{\text{Coh}(X)}^1 \Upsilon_{\text{Coh}(X)}^0(E) = \widehat{\Upsilon}_{\text{Coh}(X)}^2 \Upsilon_{\text{Coh}(X)}^3(E) = \widehat{\Upsilon}_{\text{Coh}(X)}^3 \Upsilon_{\text{Coh}(X)}^3(E) = 0$ ,  $\widehat{\Upsilon}_{\text{Coh}(X)}^0 \Upsilon_{\text{Coh}(X)}^1(E) \cong \widehat{\Upsilon}_{\text{Coh}(X)}^2 \Upsilon_{\text{Coh}(X)}^0(E)$ , and  $\widehat{\Upsilon}_{\text{Coh}(X)}^1 \Upsilon_{\text{Coh}(X)}^3(E) \cong \widehat{\Upsilon}_{\text{Coh}(X)}^3 \Upsilon_{\text{Coh}(X)}^2(E)$ .

Let  $\mathbf{R}\Delta$  denote the derived dualizing functor  $\mathbf{R}\mathcal{H}om(-, \mathcal{O})[3]$ . Let  $\tilde{\Upsilon}$  be the FMT with kernel the universal bundle  $\mathcal{E}^*$  on  $X \times X$ . As in [Muk2, (3.8)] we have the following isomorphism.

**Proposition 5.2.** ([PP, Lemma 2.2])

$$\mathbf{R}\Delta \circ \tilde{\Upsilon} \cong (\Upsilon \circ \mathbf{R}\Delta)[3]$$

This gives us the convergence of the following spectral sequence.

**“Duality” Spectral Sequence 5.3.**

$$\Upsilon_{\mathrm{Coh}(X)}^p(\mathcal{E}x t^{q+3}(E, \mathcal{O})) \implies ? \iff \mathcal{E}x t^{p+3}(\tilde{\Upsilon}_{\mathrm{Coh}(X)}^{3-q}(E), \mathcal{O})$$

for  $E \in \mathrm{Coh}(X)$ .

**Note 5.4.** Let  $\mathrm{ch}^q(E) = (a_0, a_1, a_2, a_3)$ . Then we have  $\mathrm{ch}^{-q}(\mathbf{R}\mathcal{H}om(E, \mathcal{O})) = (a_0, -a_1, a_2, -a_3)$ . Therefore for the FMT  $\tilde{\Upsilon}$  we have  $\mathrm{ch}^{w/y}(\tilde{\Upsilon}(\mathcal{O}_s)) = \mathrm{ch}^{w/y}(\mathcal{E}_{\{s\} \times X}^*) = (-y^3, 0, 0, 0)$ . So the induced transform is  $\tilde{\Upsilon}^H = \rho \begin{pmatrix} -x & y \\ z & -w \end{pmatrix}$ . Similar results for abelian surfaces have been considered in [YY, Lemma 6.18].

The following proposition generalizes a series of results in Section 4 of [MP].

**Proposition 5.5.** *We have the following:*

- (1) for  $E \in \mathrm{Coh}(X)$ 
  - (i)  $\Upsilon_{\mathrm{Coh}(X)}^0(E)$  is a reflexive sheaf,
  - (ii)  $\Upsilon_{\mathrm{Coh}(X)}^3(E) \in \mathcal{T}_{-w/y}$ ,
  - (iii)  $\Upsilon_{\mathrm{Coh}(X)}^0(E) \in \mathcal{F}_{-w/y}$ ;
- (2) for  $E \in \mathcal{T}_{x/y}$ 
  - (i)  $\Upsilon_{\mathrm{Coh}(X)}^3(E) = 0$ ,
  - (ii) if  $E \in \mathrm{Coh}^{\leq 1}(X)$  then  $\Upsilon_{\mathrm{Coh}(X)}^1(E) \in \mathcal{T}_{-w/y}$ ,
  - (iii)  $\Upsilon_{\mathrm{Coh}(X)}^2(E) \in \mathcal{T}_{-w/y}$ ;
- (3) for  $E \in \mathcal{F}_{x/y}$ 
  - (i)  $\Upsilon_{\mathrm{Coh}(X)}^0(E) = 0$ ,
  - (ii)  $\Upsilon_{\mathrm{Coh}(X)}^1(E)$  is a reflexive sheaf,
  - (iii)  $\Upsilon_{\mathrm{Coh}(X)}^1(E) \in \mathcal{F}_{-w/y}$ .

*Proof.* Proofs of (1), (2) and (3) are identical to the corresponding propositions in [MP] as listed below after replacing the Chern characters with their twisted counterparts.

- (1) (i) [MP, Proposition 4.5], (ii) and (iii) [MP, Proposition 4.7].
- (2) (i) [MP, Proposition 4.6], (ii) [MP, Proposition 4.10], (iii) [MP, Corollary 4.17].
- (3) (i) [MP, Proposition 4.6], (ii) [MP, Proposition 4.8], (iii) [MP, Proposition 4.16 (i)].

□

For  $\lambda \in \mathbb{Q}$ , let  $\mathcal{H}_\lambda$  be the abelian subcategory of  $\mathrm{Coh}(X)$  generated by stable semi-homogeneous bundles having the Chern character  $(a^3, a^2b, ab^2, b^3)$  satisfying  $\lambda = b/a$  and  $\gcd(a, b) = 1$ . Then  $\mathcal{H}_0$  consists of all homogeneous bundles on  $X$ .

Let  $H_\lambda \in \mathcal{H}_\lambda$ . The functor  $(-) \otimes H_\lambda$  is of Fourier-Mukai type with kernel  $\delta_*(H_\lambda)$  on  $X \times X$ , where  $\delta : X \rightarrow X \times X$  is the diagonal embedding. We abuse notation to write  $H_\lambda$



for the functor  $(-) \otimes H_\lambda$ . If the rank of  $H_\lambda$  is  $r$  then the functor  $H_\lambda$  induces a linear map on  $H^{2*}(X, \mathbb{Q})$  and in matrix form it is given by

$$H_\lambda^H = r \rho \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}.$$

For some integer  $n > 0$ , let  $\Phi_j$ ,  $j = 1, \dots, n+1$  be any collection of FMTs in  $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$ . For  $\lambda_i \in \mathbb{Q}$ ,  $i = 1, \dots, n$  let  $H_{\lambda_i} \in \mathcal{H}_{\lambda_i}$ . Consider the functor  $\Pi : D^b(X) \rightarrow D^b(X)$  defined by

$$(*) \quad \Pi = \Phi_{n+1} \circ H_{\lambda_n} \circ \Phi_n \circ \dots \circ H_{\lambda_2} \circ \Phi_2 \circ H_{\lambda_1} \circ \Phi_1[p].$$

Since the image of a skyscraper sheaf  $\mathcal{O}_s$  under any  $\Phi_i$  is a (shift of) a semi-homogeneous bundle and being semi-homogeneous is closed under tensoring, the image of any skyscraper sheaf  $\mathcal{O}_s$  under a composition of  $(-) \otimes H_{\lambda_i}$ s and FMTs in  $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2, \mathbb{Z})}$  has  $\mathrm{Coh}(X)$ -cohomology concentrated in one position. So we can fix  $p$  to be the unique integer for which  $\Pi_{\mathrm{Coh}(X)}^i(\mathcal{O}_s) = 0$  for  $i \neq 0$ . Then  $\Pi$  is an FM functor with kernel a sheaf  $\mathcal{U}$  on  $X \times X$ . Hence for any  $E \in \mathrm{Coh}(X)$ ,  $\Pi(E)$  can have non-trivial  $\mathrm{Coh}(X)$  cohomology at  $0, 1, 2, 3$  positions only. Also one can show that  $\Pi$  induces a linear map  $\Pi^H$  on  $H^{2*}(X, \mathbb{Q})$  given by

$$\Pi^H = a \rho \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

for some  $a \in \mathbb{Z}_{>0}$  and  $x, y, z, w \in \mathbb{Q}$  with  $xw - yz = 1$ . So  $\mathcal{U}_{\{s\} \times X} = \Pi(\mathcal{O}_s)$  has the Chern character  $a(-y^3, y^2w, -yw^3, w^3)$ . Assume  $\mathcal{U}_{\{s\} \times X}$  is not torsion, i.e.  $y < 0$ .

The functor  $\widehat{\Pi} : D^b(X) \rightarrow D^b(X)$  is defined by

$$\widehat{\Pi} = (\Phi_1)^{-1} \circ H_{\lambda_1}^* \circ (\Phi_2)^{-1} \circ \dots \circ H_{\lambda_{n-1}}^* \circ (\Phi_n)^{-1} \circ H_{\lambda_n}^* \circ (\Phi_{n+1})^{-1}[-p-3].$$

One can check that  $\widehat{\Pi}$  is an FM functor with kernel  $\Sigma^* \mathbf{R} \mathcal{H}om(\mathcal{U}, \mathcal{O})$  on  $X \times X$  (as before,  $\Sigma : X \times X \rightarrow X \times X$  switches the factors). Moreover,  $\widehat{\Pi}(\mathcal{O}_s) \in \mathrm{Coh}(X)$  for any  $s \in X$ . So the FM kernel of  $\widehat{\Pi}$  is  $\Sigma^* \mathcal{U}^*$ . Also  $\mathcal{U}$  is locally free as  $\mathcal{U}_{\{s\} \times X}$  and  $\mathcal{U}_{X \times \{s\}}^*$  are locally free. Moreover, for any  $E \in \mathrm{Coh}(X)$ ,  $\widehat{\Pi}(E)$  can have non-trivial  $\mathrm{Coh}(X)$  cohomology at  $0, 1, 2, 3$  positions only, and  $\widehat{\Pi}[3]$  is left and right adjoint to  $\Pi$  (and vice versa). The FM functor  $\widehat{\Pi}$  induces a linear map  $\widehat{\Pi}^H$  on  $H^{2*}(X, \mathbb{Q})$  given by

$$\widehat{\Pi}^H = a \rho \begin{pmatrix} -w & y \\ z & -x \end{pmatrix}.$$

We have the isomorphisms

$$\widehat{\Pi} \circ \Pi \cong H_0[-3], \quad \text{and} \quad \Pi \circ \widehat{\Pi} \cong H_0[-3]$$

for some homogeneous bundles  $H_0 \in \mathcal{H}_0$  with  $\mathcal{O}$  as a direct summand of  $H_0$ . Therefore we have the convergence of spectral sequences

$$\begin{aligned} (\dagger) \quad E_2^{p,q} &= \widehat{\Pi}_{\mathrm{Coh}(X)}^p \Pi_{\mathrm{Coh}(X)}^q(E) \implies H_{\mathrm{Coh}(X)}^{p+q-3}(H_0 E), \\ E_2^{p,q} &= \Pi_{\mathrm{Coh}(X)}^p \widehat{\Pi}_{\mathrm{Coh}(X)}^q(E) \implies H_{\mathrm{Coh}(X)}^{p+q-3}(H_0 E), \end{aligned}$$

for  $E$ . Here  $\Pi_{\mathrm{Coh}(X)}^i(-) = H_{\mathrm{Coh}(X)}^i(\Pi(-))$ .

For any  $E, F \in D^b(X)$  we have

$$\mathrm{Hom}(E, F) \hookrightarrow \mathrm{Hom}(E, H_0 F), \quad \text{since } \mathcal{O} \text{ is a direct summand of } H_0$$

$$\begin{aligned}
&\cong \operatorname{Hom}(E, \widehat{\Pi} \circ \Pi(F)[3]) \\
&\cong \operatorname{Hom}(\Pi(E), \Pi(F)), \text{ from the adjointness of } \widehat{\Pi}[3] \text{ and } \Pi.
\end{aligned}$$

Note that  $\mathbf{R} \Delta \circ H_\lambda^* \cong H_\lambda \circ \mathbf{R} \Delta$ . Therefore by iteratively using Proposition 5.2 for each of the FMTs  $\Phi_j$  together with the above isomorphism we have

$$\mathbf{R} \Delta \circ \widetilde{\Pi} \cong (\Pi \circ \mathbf{R} \Delta)[3]$$

for some FM functor  $\widetilde{\Pi}$  which is of the form  $(*)$ . Moreover, the FM kernel of  $\widetilde{\Pi}$  is  $\mathcal{U}^*$  on  $X \times X$  and the induced linear map on  $H^{2*}(X, \mathbb{Q})$  of  $\widetilde{\Pi}$  is

$$\widetilde{\Pi}^H = a \rho \begin{pmatrix} -x & y \\ z & -w \end{pmatrix}.$$

The above isomorphism involving the derived dualizing functor gives us the convergence of the spectral sequence

$$(\dagger\dagger) \quad \Pi_{\operatorname{Coh}(X)}^p(\mathcal{E}xt^{q+3}(E, \mathcal{O})) \implies ? \longleftarrow \mathcal{E}xt^{p+3}(\widetilde{\Pi}_{\operatorname{Coh}(X)}^{3-q}(E), \mathcal{O}),$$

for  $E \in \operatorname{Coh}(X)$ .

The following proposition generalizes the results on FMTs in Proposition 5.5 for FM functors of the form  $(*)$ .

**Proposition 5.6.** *We have the following:*

- (1) for  $E \in \operatorname{Coh}(X)$ 
  - (i)  $\Pi_{\operatorname{Coh}(X)}^0(E)$  is a reflexive sheaf,
  - (ii)  $\Pi_{\operatorname{Coh}(X)}^3(E) \in \mathcal{T}_{-w/y}$ ,
  - (iii)  $\Pi_{\operatorname{Coh}(X)}^0(E) \in \mathcal{F}_{-w/y}$ ;
- (2) for  $E \in \mathcal{T}_{x/y}$ 
  - (i)  $\Pi_{\operatorname{Coh}(X)}^3(E) = 0$ ,
  - (ii) if  $E \in \operatorname{Coh}^{\leq 1}(X)$  then  $\Pi_{\operatorname{Coh}(X)}^1(E) \in \mathcal{T}_{-w/y}$ ,
  - (iii)  $\Pi_{\operatorname{Coh}(X)}^2(E) \in \mathcal{T}_{-w/y}$ ;
- (3) for  $E \in \mathcal{F}_{x/y}$ 
  - (i)  $\Pi_{\operatorname{Coh}(X)}^0(E) = 0$ ,
  - (ii)  $\Pi_{\operatorname{Coh}(X)}^1(E)$  is a reflexive sheaf,
  - (iii)  $\Pi_{\operatorname{Coh}(X)}^1(E) \in \mathcal{F}_{-w/y}$ .

*Proof.* The proofs are similar to that of Proposition 5.5 or the series of similar results in Section 4 of [MP]. To illustrate the similarity, we shall give the proof for (1)(i) as follows.

Let  $s \in X$ . Then for  $0 \leq i \leq 2$ , we have

$$\begin{aligned}
\operatorname{Hom}(\mathcal{O}_s, \Pi_{\operatorname{Coh}(X)}^0(E)[i]) &\hookrightarrow \operatorname{Hom}(\widehat{\Pi}(\mathcal{O}_s), \widehat{\Pi} \Pi_{\operatorname{Coh}(X)}^0(E)[i]) \\
&\cong \operatorname{Hom}(\mathcal{U}_{X \times \{s\}}^*, \widehat{\Pi}_{\operatorname{Coh}(X)}^2 \Pi_{\operatorname{Coh}(X)}^0(E)[-2+i])
\end{aligned}$$

from the convergence of the Spectral Sequence  $(\dagger)$  for  $E$ . So  $\operatorname{Hom}(\mathcal{O}_s, \Pi_{\operatorname{Coh}(X)}^0(E)) = \operatorname{Ext}^1(\mathcal{O}_s, \Pi_{\operatorname{Coh}(X)}^0(E)) = 0$ , and

$$\begin{aligned}
\operatorname{Ext}^2(\mathcal{O}_s, \Pi_{\operatorname{Coh}(X)}^0(E)) &\hookrightarrow \operatorname{Hom}(\mathcal{U}_{X \times \{s\}}^*, \widehat{\Pi}_{\operatorname{Coh}(X)}^2 \Pi_{\operatorname{Coh}(X)}^0(E)) \\
&\cong \operatorname{Hom}(\mathcal{U}_{X \times \{s\}}^*, \widehat{\Pi}_{\operatorname{Coh}(X)}^0 \Pi_{\operatorname{Coh}(X)}^1(E)), \text{ from Spectral Sequence } (\dagger)
\end{aligned}$$

$$\begin{aligned}
&\cong \operatorname{Hom}(\widehat{\Pi}(\mathcal{O}_s), \widehat{\Pi} \Pi_{\operatorname{Coh}(X)}^1(E)), \\
&\cong \operatorname{Hom}(\mathcal{O}_s, \Pi \widehat{\Pi} \Pi_{\operatorname{Coh}(X)}^1(E)[3]), \quad \text{from the adjointness of } \widehat{\Pi}[3] \text{ and } \Pi \\
&\cong \operatorname{Hom}(\mathcal{O}_s, H_0 \Pi_{\operatorname{Coh}(X)}^1(E)).
\end{aligned}$$

Hence  $\dim\{s \in X : \operatorname{Ext}^2(\mathcal{O}_s, \Pi_{\operatorname{Coh}(X)}^0(E)) \neq 0\} \leq 0$ . Therefore  $\Pi_{\operatorname{Coh}(X)}^0(E)$  is a reflexive sheaf.  $\square$

**Proposition 5.7.** *For  $\lambda \in \mathbb{Q}_{>0}$*

- (i) *if  $E \in \operatorname{HN}_{x/y}(0, \lambda]$  then  $\Pi_{\operatorname{Coh}(X)}^0(E) \in \operatorname{HN}_{-w/y}(-\infty, -\frac{1}{2\lambda y^2}]$ ,*
- (ii) *if  $E \in \operatorname{HN}_{x/y}[-\lambda, 0]$  then  $\Pi_{\operatorname{Coh}(X)}^3(E) \in \operatorname{HN}_{-w/y}[\frac{1}{2\lambda y^2}, +\infty]$ .*

*Proof.* (i) The following proof has a similar structure to that of [MP, Proposition 4.18].

Let  $E \in \operatorname{HN}_{x/y}(0, \lambda]$ . Pick a bundle  $H_{-\lambda} \in \mathcal{H}_{-\lambda}$  of rank  $r$ . Let  $\Xi$  be the FM functor defined by

$$\Xi = \Pi \circ H_{-\lambda} \circ \widehat{\Pi}[3].$$

The induced linear map of  $\Xi$  on  $H^{2*}(X, \mathbb{Q})$  is

$$\Xi^H = -ra^2 \rho \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} -w & y \\ z & -x \end{pmatrix} = ra^2 \rho \begin{pmatrix} 1 + \lambda y w & -\lambda y^2 \\ \lambda w^2 & 1 - \lambda y w \end{pmatrix}.$$

The isomorphism  $\Xi \circ \Pi \cong \Pi \circ H_{-\lambda} \circ H_0$  gives us the convergence of spectral sequence:

$$E_2^{p,q} = \Xi_{\operatorname{Coh}(X)}^p \Pi_{\operatorname{Coh}(X)}^q(E) \implies \Pi_{\operatorname{Coh}(X)}^{p+q}(H'_{-\lambda} E)$$

for  $E$ . Here  $H'_{-\lambda} = H_{-\lambda} H_0$ . So  $H'_{-\lambda} E \in \operatorname{HN}_{x/y}(-\lambda, 0]$ . By (3)(i) of Proposition 5.6  $\Pi_{\operatorname{Coh}(X)}^0(H'_{-\lambda} E) = 0$ . Now from the convergence of the above spectral sequence, we have  $\Xi_{\operatorname{Coh}(X)}^0 \Pi_{\operatorname{Coh}(X)}^0(E) = 0$  and  $\Xi_{\operatorname{Coh}(X)}^1 \Pi_{\operatorname{Coh}(X)}^0(E) \hookrightarrow \Pi_{\operatorname{Coh}(X)}^1(H'_{-\lambda} E)$ . By (3)(iii) of Proposition 5.6,  $\Pi_{\operatorname{Coh}(X)}^1(H'_{-\lambda} E) \in \operatorname{HN}_{-w/y}(-\infty, 0]$ . Since we have  $\operatorname{HN}_{-w/y}(-\infty, 0] \subset \operatorname{HN}_{(1-\lambda y w)/\lambda y^2}(-\infty, 0]$ ,

$$(\diamond) \quad \Xi_{\operatorname{Coh}(X)}^1 \Pi_{\operatorname{Coh}(X)}^0(E) \in \operatorname{HN}_{(1-\lambda y w)/\lambda y^2}(-\infty, 0].$$

By the H-N property  $\Pi_{\operatorname{Coh}(X)}^0(E) \in \operatorname{HN}_{-w/y}(-\infty, 0]$  fits into the  $\operatorname{Coh}(X)$ -SES

$$(\clubsuit) \quad 0 \rightarrow F \rightarrow \Pi_{\operatorname{Coh}(X)}^0(E) \rightarrow G \rightarrow 0,$$

for some  $F \in \operatorname{HN}_{-w/y}(-\frac{1}{2\lambda y^2}, 0]$  and  $G \in \operatorname{HN}_{-w/y}(-\infty, -\frac{1}{2\lambda y^2}]$ . Assume  $F \neq 0$  for a contradiction. Then we can write  $\operatorname{ch}^{-w/y}(F) = (a_0, \mu a_0, a_2, a_3)$  for  $0 \geq \mu > -\frac{1}{2\lambda y^2}$ .

By applying the FM functor  $\widehat{\Pi}$  to the  $\operatorname{Coh}(X)$ -SES  $(\clubsuit)$  we have the following long exact sequence in  $\operatorname{Coh}(X)$ :

$$0 \rightarrow \widehat{\Pi}_{\operatorname{Coh}(X)}^1(G) \rightarrow \widehat{\Pi}_{\operatorname{Coh}(X)}^2(F) \rightarrow \widehat{\Pi}_{\operatorname{Coh}(X)}^2 \Pi_{\operatorname{Coh}(X)}^0(E) \rightarrow \cdots$$

By Spectral Sequence  $(\dagger)$   $\widehat{\Pi}_{\operatorname{Coh}(X)}^2 \Pi_{\operatorname{Coh}(X)}^0(E) \cong \widehat{\Pi}_{\operatorname{Coh}(X)}^0 \Pi_{\operatorname{Coh}(X)}^1(E)$  and so by (1)(iii) of Proposition 5.6 it is in  $\operatorname{HN}_{x/y}(-\infty, 0]$ . Also by (3)(iii) of Proposition 5.6  $\widehat{\Pi}_{\operatorname{Coh}(X)}^1(G) \in \operatorname{HN}_{x/y}(-\infty, 0]$ . Therefore  $\widehat{\Pi}_{\operatorname{Coh}(X)}^2(F) \in \operatorname{HN}_{x/y}(-\infty, 0]$ . By (1)(ii) of Proposition 5.6  $\widehat{\Pi}_{\operatorname{Coh}(X)}^3(F) \in \operatorname{HN}_{x/y}(0, +\infty]$ . Therefore  $\ell^2 \operatorname{ch}_1^{x/y}(\widehat{\Pi}(F)) \leq 0$ , and so  $ya_2 \leq 0$ .

Since  $\Xi_{\text{Coh}(X)}^0(F) \hookrightarrow \Xi_{\text{Coh}(X)}^0 \Pi_{\text{Coh}(X)}^0(E)$  we have  $\Xi_{\text{Coh}(X)}^0(F) = 0$ . Moreover, since  $F \in \text{HN}_{-w/y}(-\frac{1}{2\lambda y^2}, 0] = \text{HN}_{-(1+\lambda y w)/\lambda y^2}(\frac{1}{2\lambda y^2}, \frac{1}{\lambda y^2}]$  we have  $\Xi_{\text{Coh}(X)}^3(F) = 0$ . Apply the FM functor  $\Xi$  to  $\text{Coh}(X)$ -SES ( $\clubsuit$ ) and consider the long exact sequence of  $\text{Coh}(X)$ -cohomologies:

$$0 \rightarrow \Xi_{\text{Coh}(X)}^0(G) \rightarrow \Xi_{\text{Coh}(X)}^1(F) \rightarrow \Xi_{\text{Coh}(X)}^1 \Pi_{\text{Coh}(X)}^0(E) \rightarrow \cdots$$

By ( $\diamond$ ),  $\Xi_{\text{Coh}(X)}^1 \Pi_{\text{Coh}(X)}^0(E) \in \text{HN}_{(1-\lambda y w)/\lambda y^2}(-\infty, 0]$ , and by (1)(iii) of Proposition 5.6,  $\Xi_{\text{Coh}(X)}^0(G) \in \text{HN}_{(1-\lambda y w)/\lambda y^2}(-\infty, 0]$ . Therefore,  $\Xi_{\text{Coh}(X)}^1(F) \in \text{HN}_{(1-\lambda y w)/\lambda y^2}(-\infty, 0]$ . By (2)(iii) of Proposition 5.6,  $\Xi_{\text{Coh}(X)}^2(F) \in \text{HN}_{(1-\lambda y w)/\lambda y^2}(0, +\infty]$ . So

$$\ell^2 \text{ch}_1^{(1-\lambda y w)\ell/\lambda y^2}(\Xi(F)) \geq 0.$$

On the other hand, we have

$$\begin{aligned} \text{ch}^{(1-\lambda y w)/\lambda y^2}(\Xi(F)) &= r a^2 \rho \begin{pmatrix} 1 & 0 \\ \frac{1-\lambda y w}{\lambda y^2} & 1 \end{pmatrix} \begin{pmatrix} 1+\lambda y w & -\lambda y^2 \\ \lambda w^2 & 1-\lambda y w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{w}{y} & 1 \end{pmatrix} \text{ch}^{-w/y}(F) \\ &= r a^2 \rho \begin{pmatrix} 1 & -\lambda y^2 \\ \frac{1}{\lambda y^2} & 0 \end{pmatrix} \text{ch}^{-w/y}(F) \\ &= r a^2 \begin{pmatrix} * & * & * & * \\ -\frac{1}{\lambda y^2} & -2 & -\lambda y^2 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \begin{pmatrix} a_0 \\ \mu a_0 \\ a_2 \\ a_3 \end{pmatrix} \\ &= r a^2 \left( *, -2a_0 \left( \mu + \frac{1}{2\lambda y^2} \right) - \lambda^2 y^2 a_2, *, * \right). \end{aligned}$$

Here  $a_0 > 0$ ,  $\left( \mu + \frac{1}{2\lambda y^2} \right) > 0$ ,  $y a_2 \leq 0$  and so  $\ell^2 \text{ch}_1^{(1-\lambda y w)/\lambda y^2}(\Xi(F)) < 0$ . This is the required contradiction.

(ii) We shall give a proof which is similar to that of [MP, Proposition 4.19].

Let  $E \in \text{HN}_{x/y}[-\lambda, 0]$  for some  $\lambda \in \mathbb{Q}_{>0}$ . From Spectral Sequence ( $\spadesuit$ ) for  $E$  we have

$$\left( \Pi_{\text{Coh}(X)}^3(E) \right)^* \cong \tilde{\Pi}_{\text{Coh}(X)}^0(E^*).$$

Here  $\tilde{\Pi}^H = a \rho \begin{pmatrix} -x & y \\ z & -w \end{pmatrix}$  and we have  $E^* \in \text{HN}_{-x/y}[0, \lambda]$ . So by (3)(i) of Proposition 5.6 and the above result we have  $\tilde{\Pi}_{\text{Coh}(X)}^0(E^*) \in \text{HN}_{w/y}(-\infty, -\frac{1}{2\lambda y^2}]$ . Hence  $\left( \Pi_{\text{Coh}(X)}^3(E) \right)^* \in \text{HN}_{w/y}(-\infty, -\frac{1}{2\lambda y^2}]$  and so  $\Pi_{\text{Coh}(X)}^3(E) \in \text{HN}_{-w/y}[\frac{1}{2\lambda y^2}, +\infty]$  as required.  $\square$

Recall, for some fixed  $\lambda \in \mathbb{Q}_{>0}$ ,  $b = \left( \frac{x}{y} + \frac{\lambda}{2} \right)$ ,  $m = \frac{\sqrt{3}\lambda}{2}$ ,  $b' = \left( -\frac{w}{y} - \frac{1}{2\lambda y^2} \right)$  and  $m' = \frac{\sqrt{3}}{2\lambda y^2}$ . Let  $\Upsilon, \hat{\Upsilon}$  be the FMTs as introduced in subsection 4.2.

**Theorem 5.8.** *We have the following:*

- (i)  $\Upsilon(\mathcal{B}_{m\ell, b\ell}) \subset \langle \mathcal{B}_{m'\ell, b'\ell}, \mathcal{B}_{m'\ell, b'\ell}[-1], \mathcal{B}_{m'\ell, b'\ell}[-2] \rangle$ , and
- (ii)  $\hat{\Upsilon}[1](\mathcal{B}_{m'\ell, b'\ell}) \subset \langle \mathcal{B}_{m\ell, b\ell}, \mathcal{B}_{m\ell, b\ell}[-1], \mathcal{B}_{m\ell, b\ell}[-2] \rangle$ .

*Proof.* (i) If  $E \in \mathcal{F}_b = \text{HN}_{x/y}(-\infty, \frac{\lambda}{2}]$  then by (3)(i) of Proposition 5.5 and (i) of Proposition 5.7  $\Upsilon_{\text{Coh}(X)}^0(E) \in \mathcal{F}_{b'}$ . Also by (1)(ii) of Proposition 5.5  $\Upsilon_{\text{Coh}(X)}^3(E) \in \mathcal{T}_{-w/y} \subset \mathcal{T}_{b'}$ . Therefore  $\Upsilon(E)$  has  $\mathcal{B}_{m'\ell, b'\ell}$ -cohomologies in 1,2,3 positions. That is

$$\Upsilon(\mathcal{F}_b)[1] \subset \langle \mathcal{B}_{m'\ell, b'\ell}, \mathcal{B}_{m'\ell, b'\ell}[-1], \mathcal{B}_{m'\ell, b'\ell}[-2] \rangle.$$

On the other hand, if  $E \in \mathcal{T}_b = \text{HN}_{x/y}(\frac{\lambda}{2}, +\infty]$  then by (2)(i) of Proposition 5.5  $\Upsilon_{\text{Coh}(X)}^3(E) = 0$  and by (2)(iii) of Proposition 5.5  $\Upsilon_{\text{Coh}(X)}^2(E) \in \text{HN}_{-w/y}(0, +\infty] \subset \mathcal{T}_{b'}$ . So  $\Upsilon(E)$  has  $\mathcal{B}_{m'\ell, b'\ell}$ -cohomologies in positions 0,1,2 only. That is

$$\Upsilon(\mathcal{T}_b) \subset \langle \mathcal{B}_{m'\ell, b'\ell}, \mathcal{B}_{m'\ell, b'\ell}[-1], \mathcal{B}_{m'\ell, b'\ell}[-2] \rangle.$$

Hence  $\Upsilon(\mathcal{B}_{m\ell, b\ell}) \subset \langle \mathcal{B}_{m'\ell, b'\ell}, \mathcal{B}_{m'\ell, b'\ell}[-1], \mathcal{B}_{m'\ell, b'\ell}[-2] \rangle$ , as  $\mathcal{B}_{m\ell, b\ell} = \langle \mathcal{F}_b[1], \mathcal{T}_b \rangle$ .

- (ii) We can use (3)(i), (3)(iii), (2)(i), (1)(iii) of Proposition 5.5 and (ii) of Proposition 5.7 in a similar way to the above proof. □

Similar to section 5 in [MP] one can prove the following. In this case, we reduce to the special case of [MP, Theorem 5.1].

**Lemma 5.9.** *Let  $a, b \in \mathbb{Z}$  be such that  $a > 0$  and  $\gcd(a, b) = 1$ . Let  $E$  be a slope stable torsion free sheaf with  $\text{ch}_k^{b/a}(E) = 0$  for  $k = 1, 2$ . Then  $E^{**}$  is a slope stable semi-homogeneous bundle with  $\text{ch}(E^{**}) = (a^3, a^2b, ab^2, b^3)$ .*

*Proof.* The slope stable torsion free sheaf  $E$  fits into the short exact sequence  $0 \rightarrow E \rightarrow E^{**} \rightarrow T \rightarrow 0$  for some  $T \in \text{Coh}^{\leq 1}(X)$ . Now  $E^{**}$  is also slope stable and so by the usual B-G inequality  $\text{ch}_k^{b/a}(E^{**}) = 0$  for  $k = 1, 2$ . Now we have  $\text{ch}_k(\mathcal{E}nd(E^{**})) = 0$  for  $k = 1, 2$  and  $\mathcal{E}nd(E^{**})$  is a slope semistable reflexive sheaf. By [MP, Theorem 5.1]  $\mathcal{E}nd(E^{**})$  is a homogeneous bundle. Therefore  $E^{**}$  is a stable semi-homogeneous bundle (see [Muk1]) and so it is a restriction of a universal bundle which is a kernel of some FMT. Since  $X$  is principally polarized its Chern character is  $(a^3, a^2b, ab^2, b^3)$  as required. □

## 6. EQUIVALENCES OF THE CATEGORIES $\mathcal{A}_{\omega, B}$ GIVEN BY FMTs

The aim of this section is to complete the proof of Theorem 4.6.

It will be convenient to abbreviate the FMTs  $\Upsilon$  and  $\widehat{\Upsilon}[1]$  by  $\Gamma$  and  $\widehat{\Gamma}$  respectively. Then by Theorem 5.8, the images of an object from  $\mathcal{B}_{m\ell, b\ell}$  (and  $\mathcal{B}_{m'\ell, b'\ell}$ ) under  $\Gamma$  (and  $\widehat{\Gamma}$ ) are complexes whose cohomologies with respect to  $\mathcal{B}_{m'\ell, b'\ell}$  (and  $\mathcal{B}_{m\ell, b\ell}$ ) can only be non-zero in the 0, 1 or 2 positions.

The abelian category  $\mathcal{B}_{m\ell, b\ell} = \langle \mathcal{F}_b[1], \mathcal{T}_b \rangle$  does not depend on  $m > 0$ . So in the rest of the paper we write

$$\Gamma_b^i(E) := H_{\mathcal{B}_{m\ell, b\ell}}^i(\Gamma(E)).$$

We have  $\Gamma \circ \widehat{\Gamma} \cong \text{id}_{D^b(X)}[-2]$  and  $\widehat{\Gamma} \circ \Gamma \cong \text{id}_{D^b(X)}[-2]$ . This gives us the following convergence of spectral sequences and they generalize [MP, Spectral Sequence 6.1].

**Spectral Sequence 6.1.**

$$\begin{aligned} E_2^{p,q} &= \widehat{\Gamma}_b^p \Gamma_{b'}^q(E) \implies H_{\mathcal{B}_{m\ell, b\ell}}^{p+q-2}(E), \\ E_2^{p,q} &= \Gamma_{b'}^p \widehat{\Gamma}_b^q(E) \implies H_{\mathcal{B}_{m'\ell, b'\ell}}^{p+q-2}(E). \end{aligned}$$

**Proposition 6.2.** *For objects  $E$  we have the following:*

- (1) for  $E \in \mathcal{T}'_{m'\ell, b'\ell}$   
 (i)  $H^0_{\text{Coh}(X)}(\widehat{\Gamma}_b^2(E)) = 0$ , and (ii) if  $\widehat{\Gamma}_b^2(E) \neq 0$  then  $\Im Z_{m\ell, b\ell}(\widehat{\Gamma}_b^2(E)) > 0$ ,  
 (2) for  $E \in \mathcal{F}'_{m'\ell, b'\ell}$   
 (i)  $H^{-1}_{\text{Coh}(X)}(\widehat{\Gamma}_b^0(E)) = 0$ , and (ii) if  $\widehat{\Gamma}_b^0(E) \neq 0$  then  $\Im Z_{m\ell, b\ell}(\widehat{\Gamma}_b^0(E)) < 0$ ,  
 (3) for  $E \in \mathcal{T}'_{m\ell, b\ell}$   
 (i)  $H^0_{\text{Coh}(X)}(\Gamma_{b'}^2(E)) = 0$ , and (ii) if  $\Gamma_{b'}^2(E) \neq 0$  then  $\Im Z_{m'\ell, b'\ell}(\Gamma_{b'}^2(E)) > 0$ ,  
 (4) for  $E \in \mathcal{F}'_{m\ell, b\ell}$   
 (i)  $H^{-1}_{\text{Coh}(X)}(\Gamma_{b'}^0(E)) = 0$ , and (ii) if  $\Gamma_{b'}^0(E) \neq 0$  then  $\Im Z_{m'\ell, b'\ell}(\Gamma_{b'}^0(E)) < 0$ .

*Proof.* The proofs for (1), (2), (3), (4) are similar to the proofs of [MP, Propositions 6.4, 6.5, 6.6]. However we give proofs of some of them to illustrate the similarities.

(1)(i). Let  $E \in \mathcal{T}'_{m'\ell, b'\ell}$ . For any  $s \in X$ ,

$$\begin{aligned} \text{Hom}(\widehat{\Gamma}_b^2(E), \mathcal{O}_s) &\cong \text{Hom}(\widehat{\Gamma}_b^2(E), \widehat{\Gamma}_b^2(\mathcal{E}_{\{s\} \times X})) \\ &\cong \text{Hom}(\widehat{\Gamma}(E), \widehat{\Gamma}(\mathcal{E}_{\{s\} \times X})) \\ &\cong \text{Hom}(E, \mathcal{E}_{\{s\} \times X}) = 0, \end{aligned}$$

since  $E \in \mathcal{T}'_{m'\ell, b'\ell}$  and  $\mathcal{E}_{\{s\} \times X} \in \mathcal{F}'_{m'\ell, b'\ell}$ . Therefore  $H^0_{\text{Coh}(X)}(\widehat{\Gamma}_b^2(E)) = 0$  as required.

(4)(ii). Let  $E \in \mathcal{F}'_{m\ell, b\ell}$ . From (i) of (4), we have  $\Gamma_{b'}^0(E) \cong A$  for some  $0 \neq A \in \mathcal{T}_{b'} = \text{HN}_{-w/y}(-\frac{1}{2\lambda y^2}, +\infty]$ .

Consider the convergence of the spectral sequence:

$$E_2^{p,q} = \Gamma_{\text{Coh}(X)}^p(H_{\text{Coh}(X)}^q(E)) \implies \Gamma_{\text{Coh}(X)}^{p+q}(E)$$

for  $E$ . Let  $E_i = H_{\text{Coh}(X)}^i(E)$ . Then by Proposition 4.1,  $E_{-1} \in \text{HN}_{x/y}(-\infty, 0]$  and so by (3)(iii) and (1)(iii) of Proposition 5.5 we have

$$\Gamma_{\text{Coh}(X)}^1(E_{-1}) \in \text{HN}_{-w/y}(-\infty, 0], \text{ and } \Gamma_{\text{Coh}(X)}^0(E_0) \in \text{HN}_{-w/y}(-\infty, 0].$$

Therefore from the convergence of the above spectral sequence for  $E$ , we have

$$A \in \text{HN}_{-w/y}(-\frac{1}{2\lambda y^2}, +\infty] \cap \text{HN}_{-w/y}(-\infty, 0] = \text{HN}_{-w/y}(-\frac{1}{2\lambda y^2}, 0].$$

Also by (3)(ii) and (1)(i) of Proposition 5.5  $\Gamma_{\text{Coh}(X)}^1(E_{-1})$  and  $\Gamma_{\text{Coh}(X)}^0(E_0)$  are reflexive sheaves and so  $A$  is reflexive. Let  $\text{ch}^{-w/y}(A) = (a_0, a_1, a_2, a_3)$ . Then from the usual B-G inequalities for all the H-N semistable factors of  $A$  we obtain  $a_2 + \frac{1}{2\lambda y^2}a_1 \leq 0$ . So we have

$$\Im Z_{m'\ell, b'\ell}(\Gamma_{b'}^0(E)) = \Im Z_{m'\ell, b'\ell}(\Gamma_{b'}^0(A)) = \frac{3\sqrt{3}}{2\lambda y^2} \left( a_2 + \frac{1}{\lambda y^2} a_1 \right) \leq 0.$$

Equality holds when  $A \in \text{HN}_{-w/y}[0]$  with  $\text{ch}^{-w/y}(A) = (a_0, 0, 0, *)$ . Then, by considering a Jordan-Hölder filtration for  $A$  together with Lemma 5.9,  $A$  has a filtration of sheaves  $K_i$  each of them fits into the  $\text{Coh}(X)$ -SESs

$$0 \rightarrow K_i \rightarrow \mathcal{E}_{\{x_i\} \times X} \rightarrow \mathcal{O}_{Z_i} \rightarrow 0$$

for some 0-subschemes  $Z_i \subset X$ . Here  $\Gamma_{b'}^0(E) \cong A \in V_{\widehat{\mathcal{B}}_{m\ell, b\ell}}^{\widehat{\Gamma}}(2)$  implies  $A \in V_{\widehat{\text{Coh}(X)}}^{\widehat{\Gamma}}(2, 3)$ . An easy induction on the number of  $K_i$  in  $A$  shows that  $A \in V_{\widehat{\text{Coh}(X)}}^{\widehat{\Gamma}}(1, 3)$  and so  $A \in V_{\widehat{\text{Coh}(X)}}^{\widehat{\Gamma}}(3)$ .

Therefore  $Z_i = \emptyset$  for all  $i$  and so  $\widehat{\Gamma}_b^2 \Gamma_{b'}^0(E) \in \text{Coh}^0(X)$ . Now consider the convergence of the Spectral Sequence 6.1 for  $E$ . We have  $\mathcal{B}_{m\ell, b\ell}$ -SES

$$0 \rightarrow \widehat{\Gamma}_b^0 \Gamma_{b'}^1(E) \rightarrow \widehat{\Gamma}_b^2 \Gamma_{b'}^0(E) \rightarrow G \rightarrow 0,$$

where  $G$  is a subobject of  $E$  and so  $G \in \mathcal{F}'_{m\ell, b\ell}$ . Now  $\widehat{\Gamma}_b^2 \Gamma_{b'}^0(E) \in \text{Coh}^0(X) \subset \mathcal{T}'_{m\ell, b\ell}$  implies  $G = 0$  and so  $\widehat{\Gamma}_b^0 \Gamma_{b'}^1(E) \cong \widehat{\Gamma}_b^2 \Gamma_{b'}^0(E)$ . Then we have  $\Gamma_{b'}^0(E) \cong \Gamma_{b'}^0 \widehat{\Gamma}_b^0 \Gamma_{b'}^1(E) = 0$ . This is not possible as  $\Gamma_{b'}^0(E) \neq 0$ . Therefore we have the strict inequality  $\Im Z_{m'\ell, b'\ell}(\Gamma_{b'}^0(E)) < 0$  as required. This completes the proof.  $\square$

As in [MP, Lemma 6.7, Corollary 6.8, Proposition 6.9] we obtain the following table of results for the images of  $\mathcal{B}$ -objects under the FMTs.

$E$	$\Gamma_{b'}^0(E)$	$\Gamma_{b'}^1(E)$	$\Gamma_{b'}^2(E)$
$\mathcal{F}'_{m\ell, b\ell}$	0	$\mathcal{F}'_{m'\ell, b'\ell}$	$\mathcal{T}'_{m'\ell, b'\ell}$
$\mathcal{T}'_{m\ell, b\ell}$	$\mathcal{F}'_{m'\ell, b'\ell}$	$\mathcal{T}'_{m'\ell, b'\ell}$	0
$E$	$\widehat{\Gamma}_b^0(E)$	$\widehat{\Gamma}_b^1(E)$	$\widehat{\Gamma}_b^2(E)$
$\mathcal{F}'_{m'\ell, b'\ell}$	0	$\mathcal{F}'_{m\ell, b\ell}$	$\mathcal{T}'_{m\ell, b\ell}$
$\mathcal{T}'_{m'\ell, b'\ell}$	$\mathcal{F}'_{m\ell, b\ell}$	$\mathcal{T}'_{m\ell, b\ell}$	0

Now we have  $\Gamma[1](\mathcal{F}'_{m\ell, b\ell}[1]) \subset \mathcal{A}_{m'\ell, b'\ell}$  and  $\Gamma[1](\mathcal{T}'_{m\ell, b\ell}) \subset \mathcal{A}_{m'\ell, b'\ell}$ . Since  $\mathcal{A}_{m\ell, b\ell} = \langle \mathcal{F}'_{m\ell, b\ell}[1], \mathcal{T}'_{m\ell, b\ell} \rangle$ ,  $\Gamma[1](\mathcal{A}_{m\ell, b\ell}) \subset \mathcal{A}_{m'\ell, b'\ell}$ . Similarly  $\widehat{\Gamma}[1](\mathcal{A}_{m'\ell, b'\ell}) \subset \mathcal{A}_{m\ell, b\ell}$ . The isomorphisms  $\widehat{\Gamma}[1] \circ \Gamma[1] \cong \text{id}_{D^b(X)}$  and  $\Gamma[1] \circ \widehat{\Gamma}[1] \cong \text{id}_{D^b(X)}$  give us the equivalences

$$\Gamma[1](\mathcal{A}_{m\ell, b\ell}) \cong \mathcal{A}_{m'\ell, b'\ell}, \text{ and } \widehat{\Gamma}[1](\mathcal{A}_{m'\ell, b'\ell}) \cong \mathcal{A}_{m\ell, b\ell}$$

of the abelian categories as claimed in Theorem 4.6.

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